QFT Summary

LaTeX note by Ruihao Li

March 30, 2018

Contents

1 Special Relativity and Group Theory

1.1 Special relativity

Four vector notations:

$$\begin{aligned}
x^\mu &= (x^0, x^i) = (t, \mathbf{x}) \\
x_\mu &= (x^0, -x^i) = (t, -\mathbf{x}) \\
\partial^\mu &\equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \\
\partial_\mu &\equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)
\end{aligned} \tag{1.1}$$

Therefore, we have the followings:

$$\begin{aligned}
&s^2 = x_\mu x^\mu = x^\mu x_\mu = t^2 - x^2 - y^2 - z^2 \\
&\partial^2 = \partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \\
&\partial_\mu x^\nu = \partial^\mu x_\nu = \delta_{\mu\nu} \qquad (\partial_\mu x^\mu = \partial^\mu x_\mu = 4)
\end{aligned} \tag{1.2}$$

We should think of them as inner products rather than vector multiplications.
Moreover, if $\Delta s^2 < 0$, the spatial separation is greater than the distance light travels and the interval is called **space-like**. If $\Delta s^2 > 0$, the spatial separation is less than the distance light travels and the interval is called **time-like**. If $\Delta s^2 = 0$, the spatial separation is equal to the distance light travels and the interval is called **light-like**.
We can introduce a symmetric **metric tensor** $\eta_{\mu\nu} = \eta_{\nu\mu}$ defined as

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{1.3}$$

Then we have

$$s^2 = \eta^{\mu\nu} x_\mu x_\nu = \eta_{\mu\nu} x^\mu x^\nu. \tag{1.4}$$

Now we look for a set of linear transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu = \Lambda^\mu_0 x^0 + \Lambda^\mu_i x^i \tag{1.5}$$

which are Lorentz invariant (i.e. preserving s^2),

$$\begin{aligned}
\eta_{\mu\nu} &= \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho x^\sigma \\
&= \eta_{\rho\sigma} x^\rho x^\sigma.
\end{aligned} \tag{1.6}$$

Therefore, the condition for Λ to be Lorentz transformations is

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma, \tag{1.7}$$

which can also be written as

$$\hat{\eta} = \hat{\Lambda}^T \hat{\eta} \hat{\Lambda}. \tag{1.8}$$

Taking the determinant gives

$$\begin{aligned}
&\det \hat{\eta} = \det \hat{\Lambda}^T \det \hat{\eta} \det \hat{\Lambda} \\
\Longrightarrow \quad &\det \hat{\Lambda}^T \det \hat{\Lambda} = \left(\det \hat{\Lambda}\right)^2 = 1 \\
\Longrightarrow \quad &\det \hat{\Lambda} = \pm 1.
\end{aligned} \tag{1.9}$$

Proper Lorentz transformations: $\det \hat{\Lambda} = +1$.
Improper Lorentz transformations: $\det \hat{\Lambda} = -1$.
Consider the 00 component of Eq. (1.7),

$$\begin{aligned} &1 = \eta_{\mu\nu}\Lambda^{\mu}_{0}\Lambda^{\nu}_{0} = \left(\Lambda^{0}_{0}\right)^{2} - \left(\Lambda^{i}_{0}\right)^{2} \\ \implies &\left|\Lambda^{0}_{0}\right| \geqslant 1. \end{aligned} \tag{1.10}$$

Orthochronous Lorentz transformations: $\Lambda^{0}_{0} \geqslant 1$.
Non-orthochronous Lorentz transformations: $\Lambda^{0}_{0} \leqslant -1$.

In general, an object with one upper index, a^{μ} is called a **covariant vector**, while the one with one lower index, a_{μ} is called a **contravariant vector**.

1.2 Groups and representations

Definition 1. A group G is a finite or infinite set of elements which, together with an operation of multiplication, satisfies the following four fundamental properties:

1. Closure: $\forall a, b \in G,\ ab \in G$.
2. Associativity: $\forall a, b, c \in G,\ (ab)c = a(bc)$.
3. Identity: $\exists! e \in G$ such that $\forall a \in G,\ ea = ae = a$.
4. Inverse: $\forall a \in G,\ \exists! a^{-1} \in G$ such that $aa^{-1} = e$.

Definition 2. A representation ρ of a group G on a Hilbert space $\mathscr{H}$ is a mapping from G to unitary operators of $\mathscr{H}$ such that

1. $\rho(ab) = \rho(a)\rho(b)$;
2. $\rho(e) = \mathbb{1}$.

If U is a unitary transformation on $\mathscr{H}$, and we define the representation $\rho'(a) = U\rho(a)U^{\dagger}$, then $\rho'(a)$ is also a representation due the the fact that

$$\begin{aligned} &\rho'(ab) = U\rho(ab)U^{\dagger} = U\rho(a)\rho(b)U^{\dagger} = U\rho(a)U^{\dagger}U\rho(b)U^{\dagger} = \rho'(a)\rho'(b); \\ &\rho'(e) = U\rho(e)U^{\dagger} = UU^{\dagger} = \mathbb{1}. \end{aligned} \tag{1.11}$$

Therefore, $\rho'(a)$ satisfies both properties of a representation, implying that it is a representation. Moreover, $\rho(a)$ and $\rho'(a)$ are called the **equivalent representations**. Equivalent representations are related to the change of the basis in the Hilbert space $\mathscr{H}$.

1.3 The $SO(3)$and $SU(2)$ groups

The $SO(3)$ groups is a general rotation groups in 3D, whose group elements are represented by matrices that perform rotations about the three spatial axes x^1, x^2, x^3:

$$\begin{aligned} \hat{R}_{x^1}(\alpha) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \\ \hat{R}_{x^2}(\beta) &= \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \\ \hat{R}_{x^3}(\gamma) &= \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{1.12}$$

The $SO(3)$ group is non-Abelian because the group multiplication is not commutative, indeed, for instannce,

$$\hat{R}_{x^1}(\alpha)\hat{R}_{x^3}(\gamma) \neq \hat{R}_{x^3}(\gamma)\hat{R}_{x^1}(\alpha). \tag{1.13}$$

By inspecting the infinitesimal rotations, we can define the **generators** of the group which correspond to the transformation parameters. The generators in the form of differential operators are defined through their actions on a function of coordinates $F(x^1, x^2, x^3)$. For example,

$$\begin{aligned} J^3 F(x^1,x^2,x^3) &= i \lim_{\gamma\to 0}\left[\frac{F(x'^1,x'^2,x'^3)-F(x^1,x^2,x^3)}{\gamma}\right] \\ &= i \lim_{\gamma\to 0}\left[\frac{F(x^1+\gamma x^2, x^2-\gamma x^1)-F(x^1,x^2,x^3)}{\gamma}\right] \\ &=: i\left(x^2\frac{\partial}{\partial x^1}-x^1\frac{\partial}{\partial x^2}\right)F(x^1,x^2,x^3). \end{aligned} \tag{1.14}$$

Therefore,

$$J^3 = -i\left(x^1\frac{\partial}{\partial x^2}-x^2\frac{\partial}{\partial x^1}\right). \tag{1.15}$$

Similarly,

$$\begin{aligned} J^1 &= -i\left(x^2\frac{\partial}{\partial x^3}-x^3\frac{\partial}{\partial x^2}\right), \\ J^2 &= -i\left(x^3\frac{\partial}{\partial x^1}-x^1\frac{\partial}{\partial x^3}\right). \end{aligned} \tag{1.16}$$

The generators of $SO(3)$ satisfy the important commutation relation (which defines the Lie algebra)

$$\left[J^i, J^j\right] = i\epsilon^{ijk}J^k, \tag{1.17}$$

where ϵ^{ijk} is a totally antisymmetric tensor with $\epsilon^{123}=1$. The matrix expressions for the generators J^i are:

$$\begin{aligned} J^1 &= -i\frac{\mathrm{d}\hat{R}_{x^1}(\alpha)}{\mathrm{d}\alpha}\bigg|_{\alpha=0} = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \\ J^2 &= -i\frac{\mathrm{d}\hat{R}_{x^2}(\beta)}{\mathrm{d}\beta}\bigg|_{\beta=0} = \begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix} \\ J^3 &= -i\frac{\mathrm{d}\hat{R}_{x^3}(\gamma)}{\mathrm{d}\gamma}\bigg|_{\gamma=0} = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}. \end{aligned} \tag{1.18}$$

We now can write the rotation matrix for a finite transformation by exponentiating the generators

$$\hat{R}(\theta^i) = e^{iJ^i\theta^i}, \tag{1.19}$$

where θ^i are the group parameters, or in this case, the angles, and $J^i\theta^i = J^1\theta^1 + J^2\theta^2 + J^3\theta^3$. For example,

$$\begin{aligned} e^{iJ^3\gamma} &= \mathbb{1} + iJ^3\gamma - \frac{1}{2!}(J^3)^2\gamma^2 - \frac{i}{3!}(J^3)^3\gamma^3 + \cdots \\ &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \gamma\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix} + \frac{\gamma^2}{2!}\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&0\end{pmatrix} + \frac{\gamma^3}{3!}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix} + \cdots \\ &= \begin{pmatrix}\cos\gamma&\sin\gamma&0\\-\sin\gamma&\cos\gamma&0\\0&0&1\end{pmatrix} = \hat{R}_{x^3}(\gamma). \end{aligned} \tag{1.20}$$

Like the matrices of $SO(3)$ are regarded as the transformations (rotations) in 3-dimensional real space, we can view $SU(2)$ matrices as the transformations in a 2-dimensional complex space of spinors (Weyl spinors) with the following transformation properties:

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} : \ \chi' = U\chi, \ \chi'^{\dagger} = \chi^{\dagger} U^{\dagger}, \tag{1.21}$$

where U is a unitary matrix. Moreover, we can develop a correspondence between $SU(2)$ and $SO(3)$:

$$SU(2) \text{ transformations on } \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = SO(3) \text{ transformations on } \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{1.22}$$

More precisely speaking, there is a 2-to-1 correspondence $R : SU(2) \to SO(3)$ between $SU(2)$ and $SO(3)$. The map R is a group homomorphism (structure-preserving).
In analogy with rotations, the general $SU(2)$ transformations can be written as

$$U = e^{i\sigma^i \theta^i/2}. \tag{1.23}$$

Thus, we identify the generators of $SU(2)$ to be 1/2 Pauli matrices, $\frac{1}{2}\sigma^i$, which obey the commutation relations

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{ijk}\frac{\sigma^k}{2}. \tag{1.24}$$

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.25}$$

This is the 2D spinor representation of $SU(2)$ (with a basis given by Pauli matrices), used for spin-$\frac{1}{2}$ particles. We can also define the 3D vector representation of $SU(2)$, whose generators are given by

$$S^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.26}$$

The vector representation is used for spin-1 particles. In general, the dimension of the representation of $SU(2)$ is $2s+1$, where s is the spin number, i.e. the eigenvalue of the operator S^3.

2 Lorentz and Poincare Groups

2.1 The Lorentz group

Up to some possible discrete transformations, a general Lorentz transformation can be written as a product of rotations around the x, y or z axes:

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_x & \sin\theta_x \\ & & -\sin\theta_x & \cos\theta_x \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \cos\theta_y & & -\sin\theta_y \\ & & 1 & \\ & \sin\theta_y & & \cos\theta_y \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & \cos\theta_z & \sin\theta_z & \\ & -\sin\theta_z & \cos\theta_z & \\ & & & 1 \end{pmatrix} \tag{2.1}$$

and boosts in the x, y or z direction:

$$\begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sinh\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \cosh\beta_y & & \sinh\beta_y & \\ & 1 & & \\ \sinh\beta_y & & \cosh\beta_y & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} \cosh\beta_z & & & \sinh\beta_z \\ & 1 & & \\ & & 1 & \\ \sinh\beta_z & & & \cosh\beta_z \end{pmatrix}. \tag{2.2}$$

"Boosts" are just transformations which connect two inertial frames, moving with velocity $\mathbf{v}$. In the above matrices we can identify the two hyperbolic functions related to the rapidity β_i ($i = x, y, z$) as

$$\cosh\beta_i = \frac{1}{\sqrt{1 - v_i^2/c^2}} = \gamma_i, \quad \sinh\beta_i = \frac{v_i/c}{\sqrt{1 - v_i^2/c^2}} = \gamma_i \frac{v_1}{c}, \tag{2.3}$$

where γ_i are the Lorentz factors.

Matrix expressions for the generators of the boosts K^i can be computed similarly to those for the generators of rotations J^i, which are now given by 4×4 matrices. With these generators we obtain the following commutation relations:

$$\left[K^i, K^j\right] = -i\epsilon^{ijk} J^k. \tag{2.4}$$

We see that the boost transformations alone do not form a group since their generators K^i do not form a closed algebra. We also have

$$\left[K^i, J^j\right] = i\epsilon^{ijk} J^k. \tag{2.5}$$

Thus, the set of transformations composed of rotations about three axes and boosts in three directions (the Lorentz transformations) form a six-parametric group that is called the $SO(1,3)$ **Lorentz group**.

Now how do spinors transform under the Lorentz transformations? We can define the generators

$$N_+^i = \frac{1}{2}\left(J^i + iK^i\right), \qquad N_-^i = \frac{1}{2}\left(J^i - iK^i\right). \tag{2.6}$$

We can show that they satisfy the commutation relations

$$\begin{aligned} \left[N_+^i, N_-^j\right] &= 0, \\ \left[N_+^i, N_+^j\right] &= i\epsilon^{ijk} N_+^k, \\ \left[N_-^i, N_-^j\right] &= i\epsilon^{ijk} N_-^k. \end{aligned} \tag{2.7}$$

The last two commutation relations tell us that N_+^i and N_-^i can be viewed as generators of two $SU(2)$ groups. The first commutation relations says that the transformations under these two $SU(2)$ groups are independent. Therefore, the Lorentz group can be viewed as a *direct product* of two $SU(2)$ groups, $SU(2)_+ \times SU(2)_-$. We can label the states by two angular momenta (s_+, s_-) and define two types of spinor:

$$\chi \sim \left(\frac{1}{2}, 0\right) \text{ with } J^i = \frac{1}{2}\sigma^i,\ K^i = -\frac{i}{2}\sigma^i \tag{2.8}$$

and

$$\xi \sim \left(0, \frac{1}{2}\right) \text{ with } J^i = \frac{1}{2}\sigma^i,\ K^i = \frac{i}{2}\sigma^i. \tag{2.9}$$

Problem: A general Lorentz transformation $e^{i(K^i\beta^i + J^i\theta^i)}$ with either Eq. (2.8) or (2.9) as generators is not unitary!
Solution: The fundamental group for relativistic quantum systems is not the Lorentz group but the **Poincare group**.

2.2 The Poincare group

Combining the Lorentz transformations with spacetime translations we get ten-parametric transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \tag{2.10}$$

which form the Poincare group, $ISO(1,3)$. Notice that the spacetime translation does not commute with the Lorentz transformation. Hence, the Poincare group is a *semi-direct product* of the translation group P_4 and the Lorentz group $SO(1,3)$: $ISO(1,3) = P_4 \otimes SO(1,3)$.
The generators of translations are

$$P^\mu = -i\partial^\mu, \tag{2.11}$$

and the generators of Lorentz transformations are

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{2.12}$$

They satisfy the following commutation relations (Poincare algebra):

$$\begin{aligned}
&[P^\mu, P^\nu] = 0 \\
&[L^{\mu\nu}, L^{\rho\sigma}] = -i(\eta^{\mu\rho}L^{\mu\sigma} - \eta^{\mu\sigma}L^{\nu\rho} - \eta^{\nu\rho}L^{\mu\sigma} + \eta^{\nu\sigma}L^{\mu\rho}) \\
&[L^{\mu\nu}, P^\rho] = -i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu).
\end{aligned} \tag{2.13}$$

More generally, the full Lorentz spin-orbital angular momentum generators $M^{\mu\nu}$ are

$$M^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu}, \tag{2.14}$$

where the spin generators $S^{\mu\nu}$ satisfy the same commutation relations as $L^{\mu\nu}$ and $[L^{\mu\nu}, S^{\rho\sigma}] = 0$. Therefore, the generators $M^{\mu\nu}$ satisfy the same Poincare algebra (2.13). Moreover, $M^{\mu\nu}$ is an antisymmetric tensor: $M^{\mu\nu} = -M^{\nu\mu}$.

There are two Casimir operators of the Poincare group (because the Poincare group is a rank-2 group). The first one leaves $p^\mu p_\mu$ unchanged, which is given by

$$P^2 = P^\mu P_\mu, \tag{2.15}$$

where

$$P^\mu |p\rangle = p^\mu |p\rangle. \tag{2.16}$$

Consider a Lorentz transformation

$$U(\hat{\Lambda}, a)|p\rangle = \left|\hat{\Lambda}p\right\rangle, \tag{2.17}$$

with

$$P^\mu \left|\hat{\Lambda}p\right\rangle = \left(\hat{\Lambda}p\right)^\mu \left|\hat{\Lambda}p\right\rangle. \tag{2.18}$$

Then we see that

$$\left(\hat{\Lambda}p\right)^\mu \left(\hat{\Lambda}p\right)_\mu = \Lambda^\mu_\nu p^\nu \Lambda^\rho_\mu p_\rho = \delta^\rho_\nu p^\nu p_\rho = p^\nu p_\nu = p^2. \tag{2.19}$$

The second Casimir operator is

$$W^2 = W^\mu W_\mu, \tag{2.20}$$

where W^μ is the *Pauli-Lubanski pseudovector*,

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} S_{\nu\rho} P_\sigma. \tag{2.21}$$

Therefore, the Casimir invariant of the operator P^2 refers to **mass invariance**,

$$C_1 = p^2 = E^2 - |\mathbf{p}|^2 = m^2, \tag{2.22}$$

whereas the Casimir invariant of the operator W^2 refers to **spin invariance**,

$$C_2 = -m^2 s(s+1). \tag{2.23}$$

2.3 Behavior of fields under the Poincare group

A classical local field is an arbitrary function of space-time point (x^μ), which we denote by $F(x^\mu)$ in a certain reference frame. In general the functional dependence is frame-dependent, and therefore in some other reference frame the same field will be denoted by $F'(x'^\mu)$.
Consider an infinitesimal transformation which takes the initial reference frame to a primed one. The variation of the field under this transformation is

$$\begin{aligned} \delta F &= F'(x'^\mu) - F(x^\mu) \\ &= F'(x^\mu + \delta x^\mu) - F(x^\mu) \\ &= F'(x^\mu) + \partial_\mu F' \delta x^\mu + \mathcal{O}(\delta x^2) - F(x^\mu) \\ &= \underbrace{F'(x^\mu) - F(x^\mu)}_{\overline{\delta} F\ :\ \text{form variation}} + \partial_\mu F' \delta x^\mu + \mathcal{O}(\delta x^2) \end{aligned} \tag{2.24}$$

Therefore, the variation caused by the coordinate transformation can be written as

$$\delta F = \overline{\delta} F + \partial_\mu F' \delta x^\mu. \tag{2.25}$$

Scalar fields

A scalar field is the same in different inertial frames related by Lorentz transformations (LTs), i.e.

$$\phi'(x'^\mu) = \phi(x^\mu). \tag{2.26}$$

Therefore, we have the total variation of the scalar field:

$$\delta\phi = 0 = \overline{\delta}\phi + \partial_\mu \phi\, \delta x^\mu. \tag{2.27}$$

An infinitesimal LT on the spacetime coordinates can be represented by

$$\Lambda^{\mu\nu} = \delta^{\mu\nu} + \omega^{\mu\nu}, \tag{2.28}$$

where $\omega^{\mu\nu}$ is a rank-2 antisymmetric tensor which has 6 independent components with zeros on the diagonal. Hence, we can write

$$\delta x^\mu = \Lambda^{\mu\nu} x_\nu - x^\mu = \omega^{\mu\nu} x_\nu. \tag{2.29}$$

Setting a general Lorentz transformation on the scalar field as

$$\phi'(x^\mu) = e^{-\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}} \phi(x^\mu) \simeq \phi(x^\mu) - \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} \phi(x^\mu), \tag{2.30}$$

we can write the form variation as

$$\overline{\delta}\phi = \phi'(x^\mu) - \phi(x^\mu) = -\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} \phi. \tag{2.31}$$

Plugging this back into Eq. (2.27), we get

$$\begin{aligned}
&-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\phi + \partial_\mu\phi\,\omega^{\mu\nu}x_\nu = 0 \\
\Longrightarrow\quad &\frac{i}{2}M_{\mu\nu} = x_\nu\partial_\mu \\
\xRightarrow{\mu\leftrightarrow\nu}\quad &\frac{i}{2}M_{\nu\mu} = -\frac{i}{2}M_{\mu\nu} = x_\mu\partial_\nu \\
\xRightarrow{\text{subtraction}}\quad &M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) =: L_{\mu\nu} \\
\Longrightarrow\quad &S_{\mu\nu} = 0.
\end{aligned} \tag{2.32}$$

Therefore, the scalar field describes a particle with zero spin.
Note: Space inversion transformations form a discrete Z_2 group with two irreducible representations, parity even ($+1$) and parity odd (-1):

$$\phi'(x'^0, x'^i) = \pm\phi(x^0, -x^i). \tag{2.33}$$

The parity-odd scalar field is called a *pseudoscalar*.

Spinor fields

We already know that there exists two irreducible spinor representations of the Lorentz group. Hence, there are two two-component spinor fields (Weyl spinors), which transform as

$$\begin{aligned}
\psi'_R(x'^\mu) &= \hat{\Lambda}_R\psi_R(x^\mu) := \exp\left[\frac{i}{2}\sigma^i(\theta^i - i\phi^i)\right]\psi_R(x^\mu); \\
\psi'_L(x'^\mu) &= \hat{\Lambda}_L\psi_L(x^\mu) := \exp\left[\frac{i}{2}\sigma^i(\theta^i + i\phi^i)\right]\psi_L(x^\mu).
\end{aligned} \tag{2.34}$$

Notice that under space inversion, $\hat{\Lambda}_L \leftrightarrow \hat{\Lambda}_R$ and hence $\psi_L(x^\mu) \leftrightarrow \psi_R(x^\mu)$. Furthermore, we can define the Dirac spinor by combining ψ_L and ψ_R into a four-component spinor:

$$\psi(x^\mu) = \begin{pmatrix}\psi_R(x^\mu) \\ \psi_L(x^\mu)\end{pmatrix}, \tag{2.35}$$

which transforms under LTs as

$$\psi'(x'^\mu) = \begin{pmatrix}\hat{\Lambda}_R & 0 \\ 0 & \hat{\Lambda}_L\end{pmatrix}\psi(x^\mu), \tag{2.36}$$

and under space inversion as

$$\psi'(x'^0, -x^i) = \begin{pmatrix}0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}\psi(x^0, x^i) =: \gamma^0\psi(x^0, x^i). \tag{2.37}$$

The LT can also be written in a more compact form by introducing the other three γ-matrices (in the **chiral** or **Weyl representation**):

$$\gamma^i = \begin{pmatrix}0 & -\sigma^i \\ \sigma^i & 0\end{pmatrix}. \tag{2.38}$$

Eq. (2.36) then reads

$$\psi'(x'^\mu) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi(x^\mu), \tag{2.39}$$

where we have introduced

$$S^{\mu\nu} = \frac{1}{4i}[\gamma^\mu, \gamma^\nu] \tag{2.40}$$

and

$$\omega_{0i} = \phi_i, \quad \omega_{ij} = \epsilon_{ijk}\theta_k. \tag{2.41}$$

This is exactly the spinor representation of the Lorentz group. The defining algebra is the Clifford algebra, or Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\nu = 2\eta^{\mu\nu}\mathbb{1}_{4\times4}. \tag{2.42}$$

We can also define the **chirality operator** as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{2.43}$$

It forms the two projection operators,

$$\begin{aligned} P_L &= \frac{\mathbb{1} - \gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \\ P_R &= \frac{\mathbb{1} + \gamma^5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.44}$$

Apart from the Weyl representation, we can also work with other equivalent representations, which are related to each other by unitary transformations. Consider, for example, a new basis for the Dirac spinor

$$\psi(x^\mu) = \begin{pmatrix} \psi_1(x^\mu) \\ \psi_2(x^\mu) \end{pmatrix}, \tag{2.45}$$

where

$$\begin{aligned} \begin{pmatrix} \psi_1(x^\mu) \\ \psi_2(x^\mu) \end{pmatrix} &= U \begin{pmatrix} \psi_R(x^\mu) \\ \psi_L(x^\mu) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_R(x^\mu) \\ \psi_L(x^\mu) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R(x^\mu) + \psi_L(x^\mu) \\ \psi_R(x^\mu) - \psi_L(x^\mu) \end{pmatrix}. \end{aligned} \tag{2.46}$$

The γ-matrices in this representation can be obtained from those in the Weyl representation by a transformation, $U\gamma U^\dagger$. Explicitly, we have

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}. \tag{2.47}$$

This is the **standard** or **Dirac representation**.

Vector fields

Consider a vector field, $A_\mu(x^\nu)$, which under LT is written as

$$A_\mu(x) \to \Lambda^\nu_\mu A_\nu(\Lambda^{-1}x), \tag{2.48}$$

where $x^\nu \to x'^\nu = \Lambda^\nu_\mu x^\mu$.

Now we look at the full variation caused by the coordinate transformation,

$$\delta(A_\mu(x^\nu)) = \overline{\delta}(A_\mu(x^\nu)) + \partial_\rho(A_\mu(x^\nu))\delta x^\rho. \tag{2.49}$$

Since the components of the vector field form a Lorentz four-vector in the same way as the spacetime coordinates do, the total variation of the vector field should be of the same form as a variation of the spacetime coordinates (refer to Eq. (2.29)), i.e.

$$\delta A_\mu = \omega^\nu_\mu A_\nu = \frac{1}{2}\omega_{\rho\sigma}\left(\eta^\rho_\mu\eta^{\sigma\nu} - \eta^\sigma_\mu\eta^{\rho\nu}\right)A_\nu. \tag{2.50}$$

The form variation in this case is written as (compare with the case of a scalar field)

$$\begin{aligned}\overline{\delta}A_\mu &= -\frac{i}{2}\omega_{\rho\sigma}(M^{\rho\sigma})^\nu_\mu A_\nu \\ &= -\frac{i}{2}\omega_{\rho\sigma}(L^{\rho\sigma})^\nu_\mu A_\nu - \frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})^\nu_\mu A_\nu.\end{aligned} \tag{2.51}$$

Putting them back into Eq. (2.49) and canceling $\omega_{\rho\sigma}$ (note that $\delta x^\rho = \omega^{\rho\sigma}x_\sigma$):

$$\begin{aligned}&\frac{1}{2}\big(\eta^\rho_\mu\eta^{\sigma\nu} - \eta^\sigma_\mu\eta^{\rho\nu}\big)A_\nu = \underbrace{-\frac{i}{2}(L^{\rho\sigma})^\nu_\mu A_\nu}_{①}\underbrace{-\frac{i}{2}(S^{\rho\sigma})^\nu_\mu A_\nu}_{②}\underbrace{+\,\partial_\rho A_\mu x_\sigma}_{③} \\ &① = \frac{1}{2}(x^\rho\partial^\sigma - x^\sigma\partial^\rho)^\nu_\mu A_\nu = \frac{1}{2}\big[\eta^{\alpha\rho}\eta^{\beta\sigma}(x_\alpha\partial_\beta - x_\beta\partial_\alpha)\big]^\nu_\mu A_\nu = \big(-\eta^{\alpha\rho}\eta^{\beta\sigma}x_\beta\partial_\alpha\big)^\nu_\mu A_\nu \\ &③ = (x_\sigma\partial_\rho)^\nu_\mu A_\nu = (\eta^{\alpha\rho}\eta^{\beta\sigma}x_\beta\partial_\alpha)^\nu_\mu A_\nu \\ \Longrightarrow\quad &① + ③ = 0 \\ \Longrightarrow\quad &(S^{\rho\sigma})^\nu_\mu = -i\big(\eta^\rho_\mu\eta^{\sigma\nu} - \eta^\sigma_\mu\eta^{\rho\nu}\big).\end{aligned} \tag{2.52}$$

This describes the spin-1 state, so the vector field has $s = 1$.

The connection between the spinor representation and the vector representation of the Lorentz group...

3 Lagrangian and Hamiltonian Formalism

3.1 Relativistic fields: general properties

Consider a set of generic local relativistic fields $F_a(x^\mu)$. They can be Lorentz scalars, spinors, vectors, etc. We define the Lagrangian density $\mathcal{L}$ as a real functional of the fields $F_a(x^\mu)$ and their four derivatives $\partial_\nu F_a(x^\mu)$, which therefore has the form

$$\mathcal{L} = \mathcal{L}(F_a(x), \partial_\mu F_a(x)). \tag{3.1}$$

The action is defined as

$$S = \int_{t_1}^{t_2} dt \int d^3x \, \mathcal{L} = \int d^4x \, \mathcal{L}(F_a, \partial_\mu F_a). \tag{3.2}$$

We determine the equations of motion by **the principle of least action**. We vary the path, keeping the end points fixed and require $\delta S = 0$:

$$\begin{aligned} \delta S &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial F_a} \delta F_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu F_a)} \delta(\partial_\mu F_a) \right] \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial F_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_a)} \right) \right] \delta F_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu F_a)} \delta F_a \Big|_{\partial\Omega} \\ &= 0. \end{aligned} \tag{3.3}$$

The last term vanishes because because the field vanishes at infinity and thus, $\delta F_a \big|_{\partial\Omega} = 0$. Then this yields the famous Euler-Lagrange equations of motion for the field F_a,

$$\frac{\partial \mathcal{L}}{\partial F_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu F_a)} \right) = 0. \tag{3.4}$$

In the Hamiltonian formalism, we define a *canonical (conjugate) momentum* corresponding to the field $F_a(x)$:

$$\Pi_a(x) := \frac{\partial \mathcal{L}}{\partial(\partial_0 F_a)}. \tag{3.5}$$

The Hamiltonian density of the system is then

$$\mathcal{H} = \sum_a \Pi_a \partial_0 F_a - \mathcal{L}. \tag{3.6}$$

From here, we can derive two Hamilton's equations analogous to the Euler-Lagrange equation in the Lagrangian formalism. The first one follows from the Euler-Lagrange equation (3.4), noting that $\partial\mathcal{L}/\partial F_a = -\,\partial\mathcal{H}/\partial F_a$:

$$\partial_0 \Pi_a = -\frac{\partial \mathcal{H}}{\partial F_a}. \tag{3.7}$$

The second one follows from the definition of the Hamiltonian upon differentiation w.r.t. Π_a:

$$\partial_0 F_a = \frac{\partial \mathcal{H}}{\partial \Pi_a}. \tag{3.8}$$

The Hamiltonian density $\mathcal{H}$ corresponds to the energy density of the system.

3.2 Noether's theorem

There is a connection between the conservation laws of the isolated system and its symmetries, which is given by the **Noether's theorem**. Noethers theorem states that to every differentiable (continuous) symmetry generated by local actions, there corresponds to a conserved current.

Consider a system of local fields $F_a(x)$ which is described by the Lagrangian (3.1). Suppose we have N-parametric continuous transformations of the spacetime coordinates and fields, whose infinitesimal forms are written as

$$\begin{aligned}\delta x^\mu &= X^\mu_k(x)\omega^k,\\ \delta F_a(x) &= \Phi_{ak}(x)\omega^k,\end{aligned} \tag{3.9}$$

where ω^k $(k = 1, 2, \ldots, N)$ are parameters of the the infinitesimal transformations and $X^\mu_k(x)$ and $\Phi_{ak}(x)$ parameterize the variation of the coordinates and fields, respectively. We say a theory is invariant under the transformations if the action is invariant, and thus,

$$\mathcal{L}'(x')\, d^4x' = \mathcal{L}(x)\, d^4x. \tag{3.10}$$

This means that the variation is zero,

$$0 = \delta(\mathcal{L}(x)d^4x) = \underbrace{\delta(\mathcal{L}(x))d^4x}_{①} + \underbrace{\mathcal{L}(x)\delta(d^4x)}_{②}. \tag{3.11}$$

We first inspect ①, whose variation can be written as

$$\delta(\mathcal{L}(x)) = \overline{\delta}(\mathcal{L}(x)) + (\partial_\mu\mathcal{L}(x))\delta x^\mu. \tag{3.12}$$

The form variation of the Lagrangian (the 1st term) is due to the form variation of the fields,

$$\overline{\delta}F_a(x) = \delta F_a(x) - (\partial_\mu F_a(x))\delta x^\mu = [\Phi_{ak}(x) - (\partial_\mu F_a(x))X^\mu_k]\omega^k, \tag{3.13}$$

and their derivatives, which we can write

$$\overline{\delta}(\partial_\mu F_a) = \partial_\mu(\overline{\delta}F_a) \tag{3.14}$$

because the form variations always commute with the spacetime derivatives.
Therefore, the 1st term in Eq. (3.12) is

$$\begin{aligned}\overline{\delta}(\mathcal{L}(x)) &= \frac{\partial\mathcal{L}}{\partial F_a}\overline{\delta}F_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu F_a)}\overline{\delta}(\partial_\mu F_a)\\ &= \left[\cancelto{0}{\frac{\partial\mathcal{L}}{\partial F_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu F_a)}\right)}\right]\overline{\delta}F_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu F_a)}\overline{\delta}F_a\right)\\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu F_a)}[\Phi_{ak}(x) - (\partial_\mu F_a(x))X^\mu_k]\right)\omega^k.\end{aligned} \tag{3.15}$$

The first term vanishes if we apply the Euler-Lagrange equation ('vanishes on-shell'). The 2nd term in Eq. (3.12) is just

$$(\partial_\mu\mathcal{L}(x))\delta x^\mu = (\partial_\mu\mathcal{L}(x))X^\mu_k\omega^k. \tag{3.16}$$

So we have

$$① \ni \delta(\mathcal{L}(x)) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu F_a)}[\Phi_{ak}(x) - (\partial_\mu F_a(x))X^\mu_k]\right)\omega^k + (\partial_\mu\mathcal{L}(x))X^\mu_k\omega^k. \tag{3.17}$$

Now we consider ② in Eq. (3.11), where we need to evaluate the variation of the volume element (integration measure in the action), $\delta(d^4x)$ under the transformations. We have

$$d^4x' = |\det J|d^4x, \tag{3.18}$$

where J is the Jacobian of the transformations:

$$\begin{aligned}\det J :=& \det\left[\frac{\partial x'^\mu}{\partial x^\nu}\right]\\ =& \det\left[\delta^\mu_\nu + \partial_\nu(X^\mu_\nu)\omega^k\right]\\ =& 1 + \partial_\mu\big(X^\mu_\nu(x)\omega^k\big).\end{aligned} \tag{3.19}$$

Thus,

$$\text{②} \ni \delta(d^4x) = \partial_\mu\big(X^\mu_\nu(x)\omega^k\big)d^4x. \tag{3.20}$$

Finally, if we plug our ① and ② back into Eq. (3.11), we obtain the continuity equation:

$$\partial_\mu J^\mu_k = 0 \tag{3.21}$$

for N **Noether currents** J^μ_k:

$$J^\mu_k = \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu F_a)} - \mathcal{L}\delta^\mu_\nu\right]X^\nu_k - \frac{\partial\mathcal{L}}{\partial(\partial_\mu F_a)}\Phi_{ak}. \tag{3.22}$$

Knowing the currents, we can define the corresponding N Noether charges as

$$Q_k = \int_V d^3x\ J^0_k. \tag{3.23}$$

Take the time derivative of the charges:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Q_k(t) &= \int_V d^3x\ \partial_0 J^0_k \\ &= \int_V d^3x\ \left[\partial_\mu J^\mu_k - \partial_i J^i_k\right] \\ &= -\int_V d^3x\ \partial_i J^i_k \\ &= -\int_{\partial V} dS_i\ J^i_k = 0,\end{aligned} \tag{3.24}$$

where we have assumed that $J^i_k = 0$ at the boundary of the surface (i.e. no current flows out of the system and thus the system is closed).
Therefore, we indeed have N conserved quantities corresponding to N parametric symmetry transformations.

4 Classical Fields / One-Particle Wave Equations

4.1 Scalar field: The Klein-Gordon equation

Real scalar field

The relativistically invariant Lagrangian for a real scalar field is:

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi}_{\text{kinetic term}} \underbrace{-\frac{1}{2}m^2\phi^2}_{\text{mass term}}. \tag{4.1}$$

(Upon quantization in later chapters, we will indeed arrive at a particle interpretation, which describes particles with mass m).

The equation of motion for the scalar field ϕ is given by the Euler-Lagrange equation. We have

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \partial_\mu\partial^\mu\phi, \qquad \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi. \tag{4.2}$$

Therefore, the equation of motion becomes

$$(\Box + m^2)\phi = 0, \tag{4.3}$$

where $\Box \equiv \partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ is the Lorentz invariant operator called the *d'Alambertian operator*. This is known as the **Klein-Gordon equation** for scalar fields.
To solve the KG equation, we take the Fourier transform of the field:

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4}\phi(k)e^{-ik_\mu x^\mu}. \tag{4.4}$$

Plugging it into the KG equation (4.3), we obtain the relativistic relation

$$\begin{aligned} (k_0)^2 &= (k_i)^2 + m^2 \\ \Longrightarrow k_0 &= \pm\sqrt{(k_i)^2 + m^2}. \end{aligned} \tag{4.5}$$

Defining $\omega_{\mathbf{k}} = +\sqrt{(k_i)^2 + m^2}$, we can write the general solution as

$$\begin{aligned} \phi(x) &= \int \frac{d^4k}{(2\pi)^4}\delta(k^2 - m^2)\phi(k)e^{-ik_\mu x^\mu} \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{1}{\omega_{\mathbf{k}}}[\delta(k_0 - \omega_{\mathbf{k}}) + \delta(k_0 + \omega_{\mathbf{k}})]\phi(k)e^{-ik_\mu x^\mu} \\ &= \phi_+(x) + \phi_-(x), \end{aligned} \tag{4.6}$$

where

$$\phi_+(x) = \int \frac{d^3k}{(2\pi)^4 2\omega_{\mathbf{k}}} N_+\widetilde{\phi}_+(\mathbf{k})e^{-i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} \tag{4.7}$$

and

$$\phi_-(x) = \int \frac{d^3k}{(2\pi)^4 2\omega_{\mathbf{k}}} N_-\widetilde{\phi}_-(-\mathbf{k})e^{i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} \tag{4.8}$$

are called the positive and negative frequency modes of the field, respectively.
The reality condition $\phi(x) = \phi^*(x)$ implies

$$N_+^*\widetilde{\phi}_+^*(\mathbf{k}) = N_-\widetilde{\phi}_-(-\mathbf{k}) \equiv N^*\widetilde{\phi}^*(\mathbf{k}). \tag{4.9}$$

Moreover, imposing the normalization condition $N = (2\pi^{5/2})$, the general solution can be written as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\, 2\omega_{\mathbf{k}}} \Big[\widetilde{\phi}(\mathbf{k}) e^{-i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})} + \widetilde{\phi}^*(\mathbf{k}) e^{i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})}\Big]. \tag{4.10}$$

Now we define the **energy-momentum tensor** for the scalar field as

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\nu \phi) - \delta^\mu_\nu \mathcal{L}, \tag{4.11}$$

so we can calculate the Hamiltonian density and the 3-momentum density:

$$\begin{aligned} \mathcal{H} &= T^0_0 = \frac{1}{2}\big[(\partial_0\phi)^2 + (\partial_i\phi)^2 + m^2\phi^2\big] \\ \mathcal{P}_i &= T^0_i = \partial_0\phi\partial_i\phi. \end{aligned} \tag{4.12}$$

Putting Eq. (4.10) into them and integrating, we get the Hamiltonian

$$H = \int d^3x\, \mathcal{H} = \int d^3k\, \frac{1}{4}\Big[\widetilde{\phi}(\mathbf{k})\widetilde{\phi}^*(\mathbf{k}) + \widetilde{\phi}^*(\mathbf{k})\widetilde{\phi}(\mathbf{k})\Big], \tag{4.13}$$

and similarly, the 3-momentum

$$P_i = \int d^3x\, \mathcal{P}_i = \int d^3k\, \frac{k_i}{4\omega_{\mathbf{k}}}\Big[\widetilde{\phi}(\mathbf{k})\widetilde{\phi}^*(\mathbf{k}) + \widetilde{\phi}^*(\mathbf{k})\widetilde{\phi}(\mathbf{k})\Big]. \tag{4.14}$$

Complex scalar field

A complex scalar field can be written as linear combination of two real scalar fields:

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \phi^*(x) &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \end{aligned} \tag{4.15}$$

Since the Lagrangian is real, it is

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^*. \tag{4.16}$$

The Euler-Lagrange equation will give two KG equations

$$\begin{aligned} (\Box + m^2)\phi &= 0 \\ (\Box + m^2)\phi^* &= 0. \end{aligned} \tag{4.17}$$

We can solve these two KG equations in exactly the same way as in the real scalar field case. The essential difference is that in the case of a complex scalar field, the reality condition no longer holds, and hence $\widetilde{\phi}(\mathbf{k})$ and $\widetilde{\phi}(-\mathbf{k})$ are completely independent. So we write the solutions as

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\, 2\omega_{\mathbf{k}}} \Big[\widetilde{\phi}(\mathbf{k}) e^{-i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})} + \widetilde{\phi}(-\mathbf{k}) e^{i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})}\Big] \\ \phi^*(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\, 2\omega_{\mathbf{k}}} \Big[\widetilde{\phi}^*(-\mathbf{k}) e^{-i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})} + \widetilde{\phi}^*(\mathbf{k}) e^{i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})}\Big]. \end{aligned} \tag{4.18}$$

Similarly, we obtain the Hamiltonian

$$H = \int d^3k\, \frac{1}{2}\Big[\widetilde{\phi}(\mathbf{k})\widetilde{\phi}^*(\mathbf{k}) + \widetilde{\phi}^*(-\mathbf{k})\widetilde{\phi}(-\mathbf{k})\Big], \tag{4.19}$$

and the 3-momentum

$$P_i = \int d^3k\, \frac{k_i}{2\omega_{\mathbf{k}}}\Big[\widetilde{\phi}(\mathbf{k})\widetilde{\phi}^*(\mathbf{k}) + \widetilde{\phi}^*(-\mathbf{k})\widetilde{\phi}(-\mathbf{k})\Big]. \tag{4.20}$$

An extra global $U(1)$ symmetry compared to the case of a real scalar field, i.e. the Lagrangian for the complex scalar field is invariant under the transformations:

$$\phi(x) \to e^{-ie\alpha}\phi(x), \quad \phi^* \to \phi^*(x)e^{ie\alpha}. \tag{4.21}$$

Viewing α as an infinitesimal parameter, we write

$$\delta\phi = -ie\alpha\phi, \quad \delta\phi^* = ie\alpha\phi^* \tag{4.22}$$

We can calculate the Noether current from Eq. (3.22), where $\delta x^\mu = X^\mu_k \omega^k = 0$ because there is no change in the coordinates, and $\omega^k = \alpha$:

$$\begin{aligned} \delta\phi = \Phi_k\omega^k = \Phi\alpha = -ie\alpha\phi \quad &\Longrightarrow \quad \Phi = -ie\phi \\ \delta\phi^* = \Phi_k\omega^k = \Phi^*\alpha = ie\alpha\phi^* \quad &\Longrightarrow \quad \Phi^* = ie\phi^*. \end{aligned} \tag{4.23}$$

Thus, the Noether current is

$$\begin{aligned} J^\mu &= -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Phi^* \\ &= -\partial^\mu\phi^*(-ie\phi) - \partial^\mu\phi(ie\phi^*) \\ &= ie(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi). \end{aligned} \tag{4.24}$$

We see that this current is covariantly conserved, i.e. it satisfies the continuity equation

$$\partial_\mu J^\mu = 0 \quad \Longrightarrow \quad \partial_0\rho - \partial_i J^i = 0, \tag{4.25}$$

where

$$\rho = ie(\phi\partial^0\phi^* - \phi^*\partial^0\phi) \tag{4.26}$$

and

$$J^i = ie(\phi\partial^i\phi^* - \phi^*\partial^i\phi) \tag{4.27}$$

are called the *charge density* and the *current density*, respectively.
Thus the conserved charge curresponding to the global $U(1)$ symmetry is

$$\begin{aligned} Q &= ie\int d^3x\ (\phi\partial^0\phi^* - \phi^*\partial^0\phi) \\ &= e\int d^3k\ \frac{1}{2\omega_{\mathbf{k}}}\Big[\widetilde{\phi}(\mathbf{k})\widetilde{\phi}^*(\mathbf{k}) - \widetilde{\phi}^*(-\mathbf{k})\widetilde{\phi}(-\mathbf{k})\Big]. \end{aligned} \tag{4.28}$$

4.2 Spinor field: The Dirac equation

The relativistically invariant Lagrangian for the Dirac spinor field can be written as

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi. \tag{4.29}$$

The equation of motion for the field $\overline{\psi} = \psi^\dagger\gamma^0$ yields the well-known **Dirac equation**:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{4.30}$$

Similarly, the equation of motion for the field ψ yields:

$$i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = 0. \tag{4.31}$$

To find the solutions to the Dirac equation, let us work with the Dirac (standard) representation. Again, we take the Fourier transform of the spinor field:

$$\psi^\alpha(x) = \int \frac{d^4k}{(2\pi)^4}\psi^\alpha(k)e^{-ik_\mu x^\mu}. \tag{4.32}$$

We know that the Dirac spinor is a four-component bispinor in the 'spinor space', and in the standard representation, $\psi^\alpha(k)$ reads

$$\psi^\alpha(k) = \begin{pmatrix} u_A(k) \\ u_B(k) \end{pmatrix} = \begin{pmatrix} u_A^1 \\ u_A^2 \\ u_B^1 \\ u_B^2 \end{pmatrix}. \tag{4.33}$$

Then the Dirac equation (4.30) yields a system of homogeneous algebraic equations

$$\begin{pmatrix} (k_0 - m)\mathbb{1} & -\mathbf{k}\cdot\boldsymbol{\sigma} \\ \mathbf{k}\cdot\boldsymbol{\sigma} & -(k_0+m)\mathbb{1} \end{pmatrix}\begin{pmatrix} u_A(k) \\ u_B(k) \end{pmatrix} = 0. \tag{4.34}$$

The necessary and sufficient condition for the existence of non-trivial solutions of the system of equations is

$$\det\begin{pmatrix} (k_0 - m) & -\mathbf{k}\cdot\boldsymbol{\sigma} \\ \mathbf{k}\cdot\boldsymbol{\sigma} & -(k_0+m) \end{pmatrix} = 0, \tag{4.35}$$

which implies

$$k_0^2 - k_i^2 = m^2. \tag{4.36}$$

Again, let us first consider the positive frequency modes, $k_0 = \omega_{\mathbf{k}} := \sqrt{k_i^2 + m^2}$. We immediately have

$$u_B(k) = \frac{\mathbf{k}\cdot\boldsymbol{\sigma}}{\omega_{\mathbf{k}} + m}u_A(k). \tag{4.37}$$

There are two options for u_A:

$$u_A(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi_+, \qquad \text{('spin up')} \tag{4.38}$$

or

$$u_A(k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \chi_-. \qquad \text{('spin down')} \tag{4.39}$$

Thus, we have two degenerate solutions

$$u_s(\mathbf{k}) = C\begin{pmatrix} \chi_s \\ \frac{\mathbf{k}\cdot\boldsymbol{\sigma}}{\omega_{\mathbf{k}}+m}\chi_s \end{pmatrix}, \tag{4.40}$$

where s denotes the spin orientation ($+$ or $-$), and C is a normalization constant. We adopt the following normalization

$$\overline{u}_{s'}(\mathbf{k})u_s(\mathbf{k}) = 2m\delta_{s's}, \tag{4.41}$$

which gives

$$C = \sqrt{\omega_{\mathbf{k}} + m}. \tag{4.42}$$

Hence,

$$u_s(\mathbf{k}) = \sqrt{\omega_{\mathbf{k}} + m}\begin{pmatrix} \chi_s \\ \frac{\mathbf{k}\cdot\boldsymbol{\sigma}}{\omega_{\mathbf{k}}+m}\chi_s \end{pmatrix}. \tag{4.43}$$

Similarly, for the negative frequency modes, $v_s(k)$, where $k_0 = -\omega_{\mathbf{k}}$, if we adopt a slightly different normalization,

$$\overline{v}_{s'}(\mathbf{k})v_s(\mathbf{k}) = -2m\delta_{s's}, \tag{4.44}$$

we will get another two degenerate solutions

$$v_s = \sqrt{\omega_{\mathbf{k}} + m}\begin{pmatrix} \frac{-\mathbf{k}\cdot\boldsymbol{\sigma}}{\omega_{\mathbf{k}}+m}\chi_s \\ \chi_s \end{pmatrix}. \tag{4.45}$$

Then we can write down the general solution to the Dirac equation,

$$\psi(x) = \psi_+(x) + \psi_-(x), \tag{4.46}$$

where

$$\psi_+(x) = \int \frac{d^3k}{(2\pi)^{3/2}\, 2\omega_{\mathbf{k}}} \sum_s c_s(\mathbf{k}) u_s(\mathbf{k}) e^{-i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})}, \tag{4.47}$$

and

$$\psi_-(x) = \int \frac{d^3k}{(2\pi)^{3/2}\, 2\omega_{\mathbf{k}}} \sum_s d_s(-\mathbf{k}) v_s(-\mathbf{k}) e^{i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})}. \tag{4.48}$$

Note that $c_s(\mathbf{k})$ and $d_s(-\mathbf{k})$ are complex functions, while $u_s(\mathbf{k})$ and $v_s(-\mathbf{k})$ are spinors.
Similarly we have

$$\overline{\psi}(x) = \overline{\psi}_+(x) + \overline{\psi}_-(x) \tag{4.49}$$

for Dirac conjugated spinors.

Next, let us define the Hamiltonian and the 3-momentum for the Dirac field. The energy-momentum tensor (4.11) for the Dirac field is

$$\begin{aligned} T^\mu_\nu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(\partial_\nu\psi) + \cancelto{0}{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}}\left(\partial_\nu\overline{\psi}\right) - \mathcal{L}\delta^\mu_\nu \\ &= i\overline{\psi}\gamma^\mu\partial_\nu\psi - \overline{\psi}(i\gamma^\rho\partial_\rho\psi - m\psi)\delta^\mu_\nu \\ &= i\overline{\psi}\gamma^\mu\partial_\nu\psi, \end{aligned} \tag{4.50}$$

where in the second line we have used the equation of motion (the Dirac equation) so that the 2nd term vanishes. Then we have:

$$\begin{aligned} \mathcal{H} &= T^0_0 = i\psi^\dagger\partial_0\psi \\ \mathcal{P}_i &= T^0_i = i\psi^\dagger\partial_i\psi. \end{aligned} \tag{4.51}$$

Let us calculate the Hamiltonian as follows:

$$\begin{aligned}
H &= \int d^3x \, \mathcal{H} \\
&= i \int d^3x \, \psi^\dagger \partial_0 \psi \\
&= \frac{i}{4(2\pi)^3} \int d^3x \, \sum_s \sum_{s'} \int \int d^3k \, d^3k' \frac{1}{\omega_{\mathbf{k}'}\omega_{\mathbf{k}}} \\
&\quad \cdot \Big[c_s^\dagger(\mathbf{k}) u_s^\dagger(\mathbf{k}) e^{i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})} + d_s^\dagger(-\mathbf{k}) v_s^\dagger(-\mathbf{k}) e^{-i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})} \Big] \\
&\quad \cdot \Big[c_{s'}(\mathbf{k}') u_{s'}(\mathbf{k}') (-i\omega_{\mathbf{k}'}) e^{-i(\omega_{\mathbf{k}'} x^0 - \mathbf{k}'\cdot\mathbf{x})} + d_{s'}(-\mathbf{k}') v_{s'}(-\mathbf{k}') (i\omega_{\mathbf{k}'}) e^{i(\omega_{\mathbf{k}'} x^0 - \mathbf{k}'\cdot\mathbf{x})} \Big] \\
&= \frac{i}{4} \sum_s \sum_{s'} \int \int d^3k \, d^3k' \frac{1}{\omega_{\mathbf{k}'}\omega_{\mathbf{k}}} \\
&\quad \cdot \Big[(-i\omega_{\mathbf{k}'}) c_s^\dagger(\mathbf{k}) c_{s'}(\mathbf{k}') u_s^\dagger(\mathbf{k}) u_{s'}(\mathbf{k}') e^{i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})x^0} \int \frac{d^3x}{(2\pi)^3} \, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} \\
&\quad + (i\omega_{\mathbf{k}'}) d_s^\dagger(-\mathbf{k}) d_{s'}(-\mathbf{k}') v_s^\dagger(-\mathbf{k}) v_{s'}(-\mathbf{k}') e^{-i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})x^0} \int \frac{d^3x}{(2\pi)^3} \, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\
&\quad + (i\omega_{\mathbf{k}'}) c_s^\dagger(\mathbf{k}) d_{s'}(-\mathbf{k}') u_s^\dagger(\mathbf{k}) v_{s'}(-\mathbf{k}') e^{i(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'})x^0} e^{-2i\mathbf{k}'\cdot\mathbf{x}} \int \frac{d^3x}{(2\pi)^3} \, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} \\
&\quad + (-i\omega_{\mathbf{k}'}) d_s^\dagger(-\mathbf{k}) c_{s'}(\mathbf{k}') v_s^\dagger(-\mathbf{k}) u_{s'}(\mathbf{k}') e^{-i(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'})x^0} e^{2i\mathbf{k}'\cdot\mathbf{x}} \int \frac{d^3x}{(2\pi)^3} \, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \Big] \\
&\quad \left(\text{Note: } \int \frac{d^3x}{(2\pi)^3} \, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = \delta^3(\mathbf{k}-\mathbf{k}') = \delta^3(\mathbf{k}'-\mathbf{k}) = \int \frac{d^3x}{(2\pi)^3} \, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} \right) \\
&= \frac{1}{2\omega_{\mathbf{k}}} \int d^3k \, \sum_s \big[c_s^\dagger(\mathbf{k}) c_s(\mathbf{k}) - d_s^\dagger(-\mathbf{k}) d_s(-\mathbf{k}) \big],
\end{aligned} \tag{4.52}$$

where in the last line we first integrate over d^3k' (so that the delta function forces $\mathbf{k}' = \mathbf{k}$), and then use the following relations which can be obtained from Eqs. (4.43) and (4.45):

$$\begin{aligned}
&u_s^\dagger(\mathbf{k}) u_{s'}(\mathbf{k}) = 2\omega_{\mathbf{k}} \delta_{ss'} \\
&v_s^\dagger(-\mathbf{k}) v_{s'}(-\mathbf{k}) = 2\omega_{\mathbf{k}} \delta_{ss'} \\
&u_s^\dagger(\mathbf{k}) v_{s'}(-\mathbf{k}) = v_s^\dagger(-\mathbf{k}) u_{s'}(\mathbf{k}) = 2\omega_{\mathbf{k}} \delta_{ss'}.
\end{aligned} \tag{4.53}$$

Similarly, the 3-momentum is found to be

$$\begin{aligned}
P_i &= \int d^3x \, \mathcal{P}_i \\
&= i \int d^3x \, \psi^\dagger \partial_i \psi \\
&= \frac{1}{2\omega_{\mathbf{k}}} \int d^3k \, k_i \sum_s \big[c_s^\dagger(\mathbf{k}) c_s(\mathbf{k}) - d_s^\dagger(-\mathbf{k}) d_s(-\mathbf{k}) \big].
\end{aligned} \tag{4.54}$$

Like in the case of the complex scalar field, the Dirac Lagrangian also possesses a global $U(1)$ symmetry under the transformations:

$$\psi(x) \to e^{-ie\alpha} \psi(x), \quad \overline{\psi}(x) \to e^{ie\alpha} \overline{\psi}(x). \tag{4.55}$$

Using Noether's theorem, we can derive the covariantly conserved current density

$$J^\mu = e \overline{\psi} \gamma^\mu \psi, \quad \text{with} \quad \partial_\mu J^\mu = 0. \tag{4.56}$$

Then the corresponding conserved charge is

$$\begin{aligned}Q &= \int d^3x\, J^0 \\ &= \frac{e}{2}\int d^3k\, \sum_s \left[c_s^\dagger(\mathbf{k})c_s(\mathbf{k}) + d_s^\dagger(-\mathbf{k})d_s(-\mathbf{k})\right].\end{aligned} \tag{4.57}$$

Upon field quantization we will see that this quantity indeed corresponds to the electric charge of a particle (e).

4.3 Massless vector field: The Maxwell's equations

The Lagrangian for a massless vector field, which describes a particle with spin-1 (photon) is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.58}$$

where $F_{\mu\nu}$ is an antisymmetric tensor known as the *field strength*:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.59}$$

Again, using the Euler-Lagrange equation, we have

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\sigma)} &= \frac{\partial}{\partial(\partial_\rho A_\sigma)}\left[-\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)\right] \\ &= -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}(\delta^\rho_\alpha\,\delta^\sigma_\beta - \delta^\rho_\beta\,\delta^\sigma_\alpha)F_{\mu\nu} - \frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}(\delta^\rho_\mu\,\delta^\sigma_\nu - \delta^\rho_\nu\,\delta^\sigma_\mu) \\ &= -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu} + \frac{1}{4}\eta^{\mu\sigma}\eta^{\nu\rho}F_{\mu\nu} - \frac{1}{4}\eta^{\rho\alpha}\eta^{\sigma\beta}F_{\alpha\beta} + \frac{1}{4}\eta^{\rho\beta}\eta^{\sigma\alpha}F_{\alpha\beta} \\ &= -\frac{1}{2}(F^{\rho\sigma} - F^{\sigma\rho}) \\ &= -F^{\rho\sigma}\end{aligned} \tag{4.60}$$

and

$$\frac{\partial\mathcal{L}}{\partial A_\sigma} = 0. \tag{4.61}$$

Therefore, the equation of motion is

$$\partial_\rho F^{\rho\sigma} = 0. \tag{4.62}$$

We can demonstrate that the massless vector field $A^\mu = \left(A^0, A^i\right)$ actually describes the electromagnetic field, i.e. it gives the Maxwell's equations. We define the electromagnetic field strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_z & 0 \end{pmatrix}, \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_z & 0 \end{pmatrix}. \tag{4.63}$$

So we can pick up the $\mathbf{E}$ and $\mathbf{B}$ components:

$$E^k = \left(-\frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}\Phi\right)^k = -\left(\partial^0 A^k - \partial^k A^0\right) = -F^{0k} \tag{4.64}$$

and

$$B^k = (\boldsymbol{\nabla}\times\mathbf{A})^k = \epsilon^{ijk}\partial^i A^j = \frac{1}{2}\epsilon^{ijk}F^{ij}. \tag{4.65}$$

Note that $\partial_i = \partial/\partial x^i = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, while $\partial^i = -\,\partial/\partial x^i = \left(-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$ (refer to Eq. (1.1)). This is what we used in the second equal sign in Eq. (4.64). Also notice that while $A^\mu = \left(A^0, A^i\right) = (\Phi, A_x, A_y, A_z)$,

$A_\mu = (A_0, A_i) = (\Phi, -A_x, -A_y, -A_z)$ so that

$$\partial_\mu A^\mu = \partial^\mu A_\mu = \frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A}. \tag{4.66}$$

Hence, for example

$$E_k = \left(-\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\Phi\right)_k = \partial_0 A_k - \partial_k A_0 = F_{0k}, \tag{4.67}$$

which is consistent with the EM field strength tensor $F_{\mu\nu}$ in Eq. (4.63).
Then we have

- $$\boldsymbol{\nabla} \cdot \mathbf{B} = \partial^k B^k = \epsilon^{ijk} \partial^k \partial^i A^j = 0, \tag{4.68}$$

  because ϵ^{ijk} is an antisymmetric tensor, while $\partial^k \partial^i$ is symmetric.

- $$\begin{aligned} (\boldsymbol{\nabla} \times \mathbf{E})^k &= \epsilon^{ijk} \partial^i E^j \\ &= -\epsilon^{ijk} \partial^i \partial^0 A^j + \cancelto{0}{\epsilon^{ijk} \partial^i \partial^j A^0} \\ &= \partial^0 \big(\underbrace{-\epsilon^{ijk} \partial^i A^j}_{= -B^k} \big) = -\frac{\partial (\mathbf{B})^k}{\partial t}, \end{aligned} \tag{4.69}$$

  where we have used $(\boldsymbol{\nabla} \times \mathbf{A})_i \equiv \epsilon_{ijk} \partial_j A_k$ and $\epsilon^{kij} = \epsilon^{ijk}$ (an even permutation).

- $$\boldsymbol{\nabla} \cdot \mathbf{E} = \partial^k E^k = -\partial^k F^{0k} = -\partial_k F^{k0} = 0 \tag{4.70}$$

  because $F^{\mu\nu}$ is antisymmetric and $\partial^k = -\partial_k$. The last part is exactly Eq. (4.62).

- $$\begin{aligned} (\boldsymbol{\nabla} \times \mathbf{B})^k - \frac{\partial (\mathbf{E})^k}{\partial t} &= \epsilon^{ijk} \partial^i B^j - \partial^0 E^k \\ &= \frac{1}{2} \epsilon^{ijk} \partial^i \big(\epsilon^{lmk} F^{lm} \big) + \partial^0 F^{0k} \\ &= \frac{1}{2} \big(\delta^{il} \delta^{jm} - \delta^{im} \delta^{jl} \big) \partial^i F^{lm} + \partial^0 F^{0k} \\ &= \frac{1}{2} \big(\partial^l F^{lj} - \partial^m F^{jm} \big) + \partial^0 F^{0k} \\ &= \frac{1}{2} \big(\partial^l F^{lj} - \partial^l F^{jl} \big) + \partial^0 F^{0k} \qquad (m \leftrightarrow l) \\ &= \partial_0 F^{0k} - \partial_i F^{ik} \qquad (F^{jl} = -F^{lj}, \quad \partial^i F^{ik} = -\partial_i F^{ik}) \\ &= \partial_\mu F^{\mu k} = 0. \end{aligned} \tag{4.71}$$

We see that the massless vector field $A_\mu(x)$ with the Lagrangian (4.58) indeed describes the EM field in 'empty space'.

As we know, the massless vector field A_μ (photon field) should contain only two physical degrees of freedom, corresponding to two possible helicity states in the Hilbert space. However, A_μ itself contains four real degrees of freedom, implying that not all the degrees of freedom are physical. This is related to the fact that the Lagrangian for the EM field possesses a $U(1)$ **gauge symmetry** (local symmetry). Consider a shift of a given vector field A_μ by a full 4-derivative of an arbitrary function $\alpha(x)$,

$$A_\mu \to A_\mu + \partial_\mu \alpha. \tag{4.72}$$

We can check that the Lagrangian (4.58) remains invariant, for example

$$\begin{aligned}\partial^\mu A^\nu - \partial^\nu A^\mu &\to \partial^\mu (A^\nu + \partial^\nu \alpha) - \partial^\nu (A^\mu + \partial^\mu \alpha) \\ &= \partial^\mu A^\nu + \cancel{\partial^\mu \partial^\nu \alpha} - \partial^\nu A^\mu - \cancel{\partial^\nu \partial^\mu \alpha} \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu.\end{aligned} \tag{4.73}$$

As we mentioned before, this is another type of internal symmetry where the parameter of the transformation $\alpha(x)$ depends on the spacetime coordinates. This is why it is called the gauge symmetry. Now we can use the gauge freedom (arbitrariness of $\alpha(x)$) to remove the unphysical degrees of freedom in our vector field. This procedure is called the **gauge fixing**.

Coulomb (radiation) gauge

Conditions:

$$A^0 = 0, \quad \partial_i A^i = 0 \qquad (\boldsymbol{\nabla} \cdot \mathbf{A} = 0). \tag{4.74}$$

The Coulomb gauge completely removes the arbitrariness of $\alpha(x)$ and hence, A_μ is uniquely defined with only two independent degrees of freedom left. However, the Coulomb gauge is not Lorentz covariant. If a Lorentz transformation to a new inertial frame is carried out, a further gauge transformation has to be made to retain the Coulomb gauge condition. Because of this, the Coulomb gauge is not used in covariant perturbation theory, which has become standard for the treatment of relativistic quantum field theories such as quantum electrodynamics (QED). Lorentz covariant gauges such as the Lorenz gauge are usually used in these theories.
The equation of motion becomes

$$\begin{aligned}0 = \partial_\mu F^{\mu\nu} =& \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu \\ =& \partial_\mu \partial^\mu A^i - \partial^\nu \cancelto{0}{\partial_i A^i} \\ =& \Box A^i,\end{aligned} \tag{4.75}$$

i.e. $\Box A^i = 0$. This is just the KG equation for the massless vector fields $A^i(x)$! Thus, we can immediately write down the general solution in analogy to Eq. (4.10):

$$A^i(x) = \int \frac{d^3k}{(2\pi)^{3/2}\, 2\omega_{\mathbf{k}}} \sum_{\lambda=1}^{2} \epsilon^i_\lambda(\mathbf{k}) \Big[\widetilde{A}_\lambda(\mathbf{k}) e^{-i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})} + \widetilde{A}^*_\lambda(\mathbf{k}) e^{i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})} \Big], \tag{4.76}$$

where we have introduced two **polarization vectors** $\epsilon_\lambda(\mathbf{k})$ with $\lambda = 1, 2$ and $\omega_{\mathbf{k}} = |k|$ since the field is massless. The gauge condition $\partial_i A^i = 0$ gives

$$k_i \epsilon^i_\lambda(\mathbf{k}) = 0, \tag{4.77}$$

which means that for a given direction of propagation $\hat{\mathbf{k}}$, $\epsilon_\lambda(\mathbf{k})$ are in the transverse direction. This is a well-known result: the EM waves are transversely polarized.
Finally, the normalization condition for the polarization vectors is

$$\epsilon^i_{\lambda'}(\mathbf{k}) \epsilon^i_\lambda(\mathbf{k}) = \delta_{\lambda'\lambda}. \tag{4.78}$$

Lorenz gauge

Condition:

$$\partial_\mu A^\mu = 0 \qquad (\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A} = 0). \tag{4.79}$$

It only removes one out of four degrees of freedom, i.e. this gauge fixing is partial.
The equation of motion under the Lorenz gauge condition is

$$\Box A^\mu = 0. \tag{4.80}$$

The general solution is

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\,2\omega_{\mathbf{k}}} \sum_{\lambda=0}^{3} \epsilon^\mu_\lambda(\mathbf{k}) \Big[\widetilde{A}_\lambda(\mathbf{k}) e^{-i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})} + \widetilde{A}^*_\lambda(\mathbf{k}) e^{i(\omega_{\mathbf{k}} x^0 - \mathbf{k}\cdot\mathbf{x})}\Big]. \tag{4.81}$$

The following normalization condition (orthonormality) is adopted:

$$\epsilon^\mu_\lambda(\mathbf{k})\epsilon_{\mu\lambda'}(\mathbf{k}) = \eta_{\lambda\lambda'}. \tag{4.82}$$

We see that $\epsilon^\mu_0(\mathbf{k})$ is a time-like 4-vector since $\epsilon^\mu_0(\mathbf{k})\epsilon_{\mu 0}(\mathbf{k}) = +1$, while $\epsilon^\mu_{1,2,3}(\mathbf{k})$ are space-like 4-vector since $\epsilon^\mu_{1,2,3}(\mathbf{k})\epsilon_{\mu 1,2,3}(\mathbf{k}) = -1$. Note that the Lorenz gauge condition gives

$$\sum_{\lambda=0}^{3} k_\mu \epsilon^\mu_\lambda(\mathbf{k}) = 0. \tag{4.83}$$

Consider a reference frame where photon is moving along the third axis so that its 4-momentum is $k^\mu = (\omega_{\mathbf{k}}, 0, 0, \omega_{\mathbf{k}})$, and the polarization vectors can be written as

$$\underbrace{\epsilon^\mu_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{\text{scalar photons (unphysical)}}, \quad \underbrace{\epsilon^\mu_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \epsilon^\mu_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}}_{\text{transverse photons (physical)}}, \quad \underbrace{\epsilon^\mu_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{\text{longitudinal photons (unphysical)}}. \tag{4.84}$$

Only $\epsilon^\mu_{1,2}$ are physical because they satisfy Eq. (4.83)

$$k^\mu \epsilon_{\mu 1,2} = 0. \tag{4.85}$$

Now let us calculate the Hamiltonian and the 3-momentum in the Lorenz gauge. The Lagrangian under the Lorenz gauge condition can be written as

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu) \\
&= -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \cancel{\partial_\mu A^\mu \partial^\nu A_\nu}) \qquad \text{(use Lorenz condition)} \\
&= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu.
\end{aligned} \tag{4.86}$$

So we have

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} &= -\frac{1}{2}\frac{\partial}{\partial(\partial_0 A_\mu)}(\partial_\rho \partial_\sigma \partial^\rho \partial^\sigma) \\
&= -\frac{1}{2}\eta^{\rho\alpha}\eta^{\sigma\beta}\frac{\partial}{\partial(\partial_0 A_\mu)}(\partial_\rho A_\sigma \partial_\alpha \partial_\beta) \\
&= -\frac{1}{2}\eta^{\rho\alpha}\eta^{\sigma\beta}(\delta_{0\rho}\delta_{\mu\sigma}\partial_\alpha A_\beta + \partial_\rho A_\sigma \delta_{0\alpha}\delta_{\mu\beta}) \\
&= -\frac{1}{2}\big(\eta^{0\alpha}\eta^{\mu\beta}\partial_\alpha A_\beta + \eta^{\rho 0}\eta^{\sigma\mu}\partial_\rho A_\sigma\big) \\
&= -\frac{1}{2}\big(\partial^0 A^\mu + \partial^0 A^\mu\big) = -\partial^0 A^\mu.
\end{aligned} \tag{4.87}$$

Then the Hamiltonian and 3-momentum densities are given as

$$\begin{aligned}
\mathcal{H} \equiv T^0_0 &= \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}(\partial_0 A_\mu) - \mathcal{L} \\
&= -\partial^0 A^\mu \partial_0 A_\mu + \frac{1}{2}\partial_\nu A_\mu \partial^\nu A^\mu \\
&= -\partial_0 A_\mu \partial^0 A^\mu + \frac{1}{2}\left(\partial_0 A_\mu \partial^0 A^\mu + \partial_i A_\mu \partial^i A^\mu\right) \\
&= \frac{1}{2}\left(\partial_i A_\mu \partial^i A^\mu - \partial_0 A_\mu \partial^0 A^\mu\right) \\
\mathcal{P}_k \equiv T^0_k &= \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}(\partial_k A_\mu) \\
&= -\partial_k A_\mu \partial^0 A^\mu .
\end{aligned} \tag{4.88}$$

Taking the solution (4.81) with the normalization condition (4.82) for the polarization vectors, we can calculate the Hamiltonian in the usual way and obtain

$$\begin{aligned}
H &= \int d^3x\, \mathcal{H} \\
&= \int d^3k\, \frac{1}{4}\left[\sum_{\lambda=1}^{3}\Big(\widetilde{A}_\lambda(\mathbf{k})\widetilde{A}^*_\lambda(\mathbf{k}) + \widetilde{A}^*_\lambda(\mathbf{k})\widetilde{A}_\lambda(\mathbf{k})\Big) - \Big(\widetilde{A}_0(\mathbf{k})\widetilde{A}^*_0(\mathbf{k}) + \widetilde{A}^*_0(\mathbf{k})\widetilde{A}_0(\mathbf{k})\Big)\right]
\end{aligned} \tag{4.89}$$

We see that the scalar photons' contribution to the total energy is negative and thus, the energy is not positive definite. However, actually the Lorenz condition evaluated on the solution (4.81) gives

$$\widetilde{A}_0(\mathbf{k}) = \widetilde{A}_3(\mathbf{k}) \tag{4.90}$$

so that the contribution from the unphysical states, the scalar photons and the longitudinal photons, cancel each other in the Hamiltonian and what is left is

$$H = \int d^3k\, \frac{1}{4}\sum_{\lambda=1}^{2}\Big(\widetilde{A}_\lambda(\mathbf{k})\widetilde{A}^*_\lambda(\mathbf{k}) + \widetilde{A}^*_\lambda(\mathbf{k})\widetilde{A}_\lambda(\mathbf{k})\Big). \tag{4.91}$$

Now the Hamiltonian is positive definite, as it should be.
For the 3-momentum, we get

$$P_i = \int d^3k\, \frac{k_i}{4\omega_\mathbf{k}}\sum_{\lambda=1}^{2}\Big(\widetilde{A}_\lambda(\mathbf{k})\widetilde{A}^*_\lambda(\mathbf{k}) + \widetilde{A}^*_\lambda(\mathbf{k})\widetilde{A}_\lambda(\mathbf{k})\Big). \tag{4.92}$$

Finally, an important lesson: all gauge fixings are physically equivalent. i.e. if we do the same calculations again in the Coulomb gauge, we will get the same results.

5 Canonical Quantization of Scalar Fields

5.1 Real scalar fields

As we have seen previously, the Lagrangian for a real scalar field is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{5.1}$$

from which we can define the conjugate momentum as

$$\Pi(x) := \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi. \tag{5.2}$$

The Hamiltonian density is then

$$\mathcal{H} := \Pi(x)\partial_0\phi - \mathcal{L} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{5.3}$$

Upon quantization, we promote the field $\phi(x)$ and the conjugate momentum $\Pi(x)$ to operators $\hat{\phi}(x)$ and $\hat{\Pi}(x)$ (we will just write the operators without the hat from now on). In free quantum theory, we usually work in the Heisenberg picture, where the field operators carry the time dependence. Then we impose the equal-time canonical commutation relations:

$$\begin{aligned}
\big[\phi(x^0,\mathbf{x}),\Pi(x^0,\mathbf{x}')\big] &= i\delta^3(\mathbf{x}-\mathbf{x}'),\\
\big[\phi(x^0,\mathbf{x}),\phi(x^0,\mathbf{x}')\big] &= 0,\\
\big[\Pi(x^0,\mathbf{x}),\Pi(x^0,\mathbf{x}')\big] &= 0.
\end{aligned} \tag{5.4}$$

The two fields obey the Heisenberg equation[1]:

$$i\partial_0\phi = [\phi, H], \quad i\partial_0\Pi = [\Pi, H], \tag{5.5}$$

where H is the quantum Hamiltonian, written as

$$H = \frac{1}{2}\int d^3x\,\big(\Pi^2 + \partial_i\phi\partial_i\phi + m^2\phi\phi\big). \tag{5.6}$$

It is straightforward to show that

$$\begin{aligned}
\partial_0\partial_0\phi &= -i[\partial_0\phi, H] \qquad \text{(from the Heisenberg equation)}\\
&= -i[\Pi, H] = \partial_0\Pi \qquad \text{(from the Heisenberg equation)}\\
&= \big(\partial_i\partial_i - m^2\big)\phi. \qquad \text{(from Eq. (5.6))}
\end{aligned} \tag{5.7}$$

Therefore, the field operator ϕ indeed obeys the KG equation, $(\Box + m^2)\phi = 0$, as the classical field. If we replace the Fourier images in Eq. (4.10) by the following operators:

$$\frac{1}{\sqrt{2\omega_\mathbf{k}}}\widetilde{\phi}(\mathbf{k}) \to a(\mathbf{k}), \qquad \frac{1}{\sqrt{2\omega_\mathbf{k}}}\widetilde{\phi}^*(\mathbf{k}) \to a^\dagger(\mathbf{k}), \tag{5.8}$$

the general solution can be written as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{k}}}\Big[a(\mathbf{k})e^{-i(\omega_\mathbf{k}x^0-\mathbf{k}\cdot\mathbf{x})} + a^\dagger(\mathbf{k})e^{i(\omega_\mathbf{k}x^0-\mathbf{k}\cdot\mathbf{x})}\Big]. \tag{5.9}$$

The conjugate momentum is written as

$$\Pi(x) = \partial_0\phi(x) = \int \frac{d^3k\ (i\omega_\mathbf{k})}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{k}}}\Big[-a(\mathbf{k})e^{-i(\omega_\mathbf{k}x^0-\mathbf{k}\cdot\mathbf{x})} + a^\dagger(\mathbf{k})e^{i(\omega_\mathbf{k}x^0-\mathbf{k}\cdot\mathbf{x})}\Big] \tag{5.10}$$

[1] See the discussion on the Schrodinger picture and the Heisenberg picture in Appendix.

To obtain the commutation relations for the annihilation operator $a(\mathbf{k})$ and the creation operator $a^\dagger(\mathbf{k})$, we take the *partial Fourier transform* of the field operator and its conjugate momentum:

$$\begin{aligned}\phi(x^0,\mathbf{k}) &= \int \frac{d^3x}{(2\pi)^3}\phi(x)e^{-i\mathbf{k}\cdot\mathbf{x}} \\ \Pi(x^0,\mathbf{k}) &= \int \frac{d^3x}{(2\pi)^3}\Pi(x)e^{-i\mathbf{k}\cdot\mathbf{x}}\end{aligned} \tag{5.11}$$

and use the integral form of the δ-function:

$$\delta^3(\mathbf{k}-\mathbf{k}') = \int \frac{d^3x}{(2\pi)^3}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}. \tag{5.12}$$

Then we have (plugging Eqs. (5.9) and (5.10) into Eq. (5.11))

$$\phi(x^0,\mathbf{k}) = \frac{1}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{k}}}\left[a(\mathbf{k})e^{-i\omega_\mathbf{k}x^0} + a^\dagger(-\mathbf{k})e^{i\omega_\mathbf{k}x^0}\right] \tag{5.13}$$

and

$$\Pi(x^0,\mathbf{k}) = \frac{i\omega_\mathbf{k}}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{k}}}\left[-a(\mathbf{k})e^{-i\omega_\mathbf{k}x^0} + a^\dagger(-\mathbf{k})e^{i\omega_\mathbf{k}x^0}\right]. \tag{5.14}$$

Hence, from the two equation above we have

$$a(\mathbf{k}) = \int \frac{d^3x}{(2\pi)^{3/2}}\sqrt{\frac{\omega_\mathbf{k}}{2}}\left[\phi(x) + \frac{i}{\omega_\mathbf{k}}\Pi(x)\right]e^{i\omega_\mathbf{k}x^0 - i\mathbf{k}\cdot\mathbf{x}} \tag{5.15}$$

and

$$a^\dagger(\mathbf{k}) = \int \frac{d^3x}{(2\pi)^{3/2}}\sqrt{\frac{\omega_\mathbf{k}}{2}}\left[\phi(x) - \frac{i}{\omega_\mathbf{k}}\Pi(x)\right]e^{-i\omega_\mathbf{k}x^0 + i\mathbf{k}\cdot\mathbf{x}}. \tag{5.16}$$

Now we can compute the following commutator:

$$\begin{aligned}\left[a(\mathbf{k}),a^\dagger(\mathbf{k}')\right] &= \int \frac{d^3x\,d^3x'}{(2\pi)^3}\frac{\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}}{2}\left[\left(\phi(x^0,\mathbf{x}) + \frac{i}{\omega_\mathbf{k}}\Pi(x^0,\mathbf{x})\right),\left(\phi(x^0,\mathbf{x}') - \frac{i}{\omega_\mathbf{k}}\Pi(x^0,\mathbf{x}')\right)\right] \\ &\quad e^{i(\omega_\mathbf{k}-\omega_{\mathbf{k}'})x^0}e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'} \\ &= \int \frac{d^3x\,d^3x'}{(2\pi)^3}\frac{\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}}{2}\left\{\frac{1}{\omega_{\mathbf{k}'}}\underbrace{\left[\Pi(x^0,\mathbf{x}'),\phi(x^0,\mathbf{x})\right]}_{=-i\delta^3(\mathbf{x}'-\mathbf{x})} + \frac{1}{\omega_\mathbf{k}}\underbrace{\left[\Pi(x^0,\mathbf{x}),\phi(x^0,\mathbf{x}')\right]}_{=-i\delta^3(\mathbf{x}-\mathbf{x}')}\right\} \\ &\quad e^{i(\omega_\mathbf{k}-\omega_{\mathbf{k}'})x^0}e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'} \\ &= \int \frac{d^3x}{(2\pi)^3}\frac{\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}}{2}\left(\frac{1}{\omega_{\mathbf{k}'}} + \frac{1}{\omega_\mathbf{k}}\right)e^{i(\omega_\mathbf{k}-\omega_{\mathbf{k}'})x^0}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\ &= \frac{\sqrt{\omega_\mathbf{k}^2}}{2}\left(\frac{2}{\omega_\mathbf{k}}\right)e^{i(\omega_\mathbf{k}-\omega_\mathbf{k})x^0}\int \frac{d^3x}{(2\pi)^3}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \qquad \text{(assuming } \omega_{\mathbf{k}'} = \omega_\mathbf{k}\text{)} \\ &= \delta^3(\mathbf{k}-\mathbf{k}').\end{aligned} \tag{5.17}$$

Similarly we obtain the other two commutation relations

$$\left[a(\mathbf{k}),a(\mathbf{k}')\right] = \left[a^\dagger(\mathbf{k}),a^\dagger(\mathbf{k}')\right] = 0. \tag{5.18}$$

5.2 Zero point energy and normal ordering

We first look at the set of *single-particle states*. The ground state (with minimum energy) is defined as the state annihilated by all the annihilation operators:

$$a(\mathbf{k})\left|0\right\rangle = 0, \quad \forall\,\mathbf{k}. \tag{5.19}$$

The one-particle state with momentum $\mathbf{k}$ can be created from the the ground state by a creation operator:

$$|\mathbf{k}\rangle = a^\dagger(\mathbf{k}) |0\rangle . \tag{5.20}$$

Using the commutation relation (5.17), we can write the norm between two states as

$$\begin{aligned} \langle \mathbf{k}'|\mathbf{k}\rangle &= \langle 0|a(\mathbf{k}')a^\dagger(\mathbf{k})|0\rangle \\ &= \langle 0|a^\dagger(\mathbf{k})a(\mathbf{k})|0\rangle + \delta^3(\mathbf{k}-\mathbf{k}') \langle 0|0\rangle \\ &= \delta^3(\mathbf{k}-\mathbf{k}'), \end{aligned} \tag{5.21}$$

where we use Eq. (5.19) and the normalization condition: $\langle 0|0\rangle = 1$ in the last line. What we obtain above is exactly the **orthonormality condition**.
The corresponding **completeness condition** is

$$\int d^3k \, |\mathbf{k}\rangle \langle \mathbf{k}| = 1. \tag{5.22}$$

We can continuously apply creation operators $a^\dagger(\mathbf{k}_1), a^\dagger(\mathbf{k}_2), \cdots$ to create *multi-particle states*:

$$|\mathbf{k}_1, \mathbf{k}_2, \cdots\rangle = a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\cdots|0\rangle . \tag{5.23}$$

A general state could have multiple applications of creation operators with the same momenta, in which case it is convenient to normalize the state with a symmetry factor:

$$|\{n_{\mathbf{k}_i}\}\rangle := |n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \cdots\rangle = |\underbrace{\mathbf{k}_1, \cdots, \mathbf{k}_1}_{n_{\mathbf{k}_1}}, \underbrace{\mathbf{k}_2, \cdots, \mathbf{k}_2}_{n_{\mathbf{k}_2}}, \cdots\rangle = \prod_i \frac{\left[a^\dagger(\mathbf{k}_i)\right]^{n_{\mathbf{k}_i}}}{\sqrt{n_{\mathbf{k}_i}}} |0\rangle , \tag{5.24}$$

where $n_{\mathbf{k}_i}$ is the number of times the creation operator with label $\mathbf{k}_i$ is applied. The Hilbert space spanned by the set of all possible states is called the **Fock space**. The basis states are interpreted as multi-particle states.
Define a new set of operators by

$$n(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}). \tag{5.25}$$

When acting on the ground state, it gives

$$n(\mathbf{k}) |0\rangle = a^\dagger(\mathbf{k})a(\mathbf{k}) |0\rangle = 0. \tag{5.26}$$

When acting on one-particle state, it gives

$$\begin{aligned} n(\mathbf{k}) |\mathbf{k}'\rangle &= a^\dagger(\mathbf{k})a(\mathbf{k}) |\mathbf{k}'\rangle \\ &= a^\dagger(\mathbf{k})a(\mathbf{k})a^\dagger(\mathbf{k}') |0\rangle \\ &= a^\dagger(\mathbf{k})\left[a^\dagger(\mathbf{k}')a(\mathbf{k}) + \delta^3(\mathbf{k}-\mathbf{k}')\right] |0\rangle \\ &= \delta^3(\mathbf{k}-\mathbf{k}')a^\dagger(\mathbf{k}) |0\rangle \\ &= \delta^3(\mathbf{k}-\mathbf{k}') |\mathbf{k}\rangle , \end{aligned} \tag{5.27}$$

where we have used the commutation relation to change the order of $a(\mathbf{k})$ and $a^\dagger(\mathbf{k}')$. We can proceed with n-particle states and observe that $n(\mathbf{k})$ serves as a *particle number density operator*. Thus, the *particle number operator*, which tells us the total number of particles in a particular state, can be defined as

$$N = \int d^3k \, n(\mathbf{k}) = \int d^3k \, a^\dagger(\mathbf{k})a(\mathbf{k}). \tag{5.28}$$

For example,

$$\begin{aligned} N\left|0\right\rangle &= 0 \\ N\left|\mathbf{k}'\right\rangle &= \int d^3k\, n(\mathbf{k})\left|\mathbf{k}'\right\rangle = 1\left|\mathbf{k}'\right\rangle \\ N\left|\{n_{\mathbf{k}_i}\}\right\rangle &= \sum_i n_{\mathbf{k}_i}\left|\{n_{\mathbf{k}_i}\}\right\rangle . \end{aligned} \tag{5.29}$$

Energy of the states can be computed from the quantum Hamiltonian operator Eq. (5.6), which can be expressed in terms of creation and annihilation operators using the expansions of the field and conjugate momentum operators Eqs. (5.9) and (5.10):

$$\begin{aligned} H &= \int d^3k\, \frac{\omega_{\mathbf{k}}}{2}\left[a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})\right] \\ &= \int d^3k\, \frac{\omega_{\mathbf{k}}}{2}\left[2a^\dagger(\mathbf{k})a(\mathbf{k}) + \delta^3(0)\right]. \end{aligned} \tag{5.30}$$

We see that we run into a problem when we try to change the order of operators because the integral of $\delta^3(0)$ is divergent (we are working in a infinite spatial volume)! Let the Hamiltonian operator act on the ground state

$$H\left|0\right\rangle = \int d^3k\, \frac{\omega_{\mathbf{k}}}{2}\left[2a^\dagger(\mathbf{k})a(\mathbf{k}) + \delta^3(0)\right]\left|0\right\rangle = \int d^3k\, \frac{\omega_{\mathbf{k}}}{2}\delta^3(0)\left|0\right\rangle , \tag{5.31}$$

and thus, we can interpret $\int d^3k\, \frac{\omega_{\mathbf{k}}}{2}\delta^3(0)$ as the ground energy, which is infinite. However, the physical argument is that the absolute values of energy have no physical meaning, only the energy difference is the quantity which we can measure in experiments. So we can define a new, **renormalized** (physical) Hamiltonian:

$$\begin{aligned} H_{\text{ren}} &= H - \int d^3k\, \frac{\omega_{\mathbf{k}}}{2}\delta^3(0) \\ &= \int d^3k\, \omega_{\mathbf{k}} a^\dagger(\mathbf{k})a(\mathbf{k}) \\ &= \int d^3k\, \omega_{\mathbf{k}} n(\mathbf{k}), \end{aligned} \tag{5.32}$$

with $H_{\text{ren}}\left|0\right\rangle = 0$.

We can subtract off the ground state energies automatically by a procedure known as the **normal ordering**, which is to place all the creation operators to the left, and all the annihilation operators to the right. For example,

$$\mathrm{N}\left[a(\mathbf{k})a^\dagger(\mathbf{k})\right] = a^\dagger(\mathbf{k})a(\mathbf{k}). \tag{5.33}$$

Hence, $\mathrm{N}[H] = H_{\text{ren}}$.

We can compute the energy of a one-particle state:

$$\begin{aligned} H_{\text{ren}}\left|\mathbf{k}\right\rangle &= \int d^3k'\, \omega_{\mathbf{k}'} n(\mathbf{k}')\left|\mathbf{k}\right\rangle \\ &= \int d^3k'\, \omega_{\mathbf{k}'}\delta^3(\mathbf{k}'-\mathbf{k})\left|\mathbf{k}'\right\rangle \\ &= \omega_{\mathbf{k}}\left|\mathbf{k}\right\rangle , \end{aligned} \tag{5.34}$$

where in the second line we have used Eq. (5.27). Hence, the energy of the one-particle state $|\mathbf{k}\rangle$ is $\omega_{\mathbf{k}}$. Recall that $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ is the energy of a relativistic particle of mass m and momentum $\mathbf{k}$.

Similarly, the momentum operator is

$$P_k = \int d^3k\, k_k n(\mathbf{k}). \tag{5.35}$$

When acting on a single-particle state $|\mathbf{k}\rangle$, it gives

$$P_k\left|\mathbf{k}\right\rangle = \int d^3k'\, k_k n(\mathbf{k}')\left|\mathbf{k}\right\rangle = k_k\left|\mathbf{k}\right\rangle . \tag{5.36}$$

5.3 Complex scalar fields and antiparticles

To quantize a complex scalar field, we replace the Fourier images in the solutions (4.18) to the two classical KG equations (4.17) by the operators (remember that $\widetilde{\phi}(\mathbf{k})$ and $\widetilde{\phi}(-\mathbf{k})$ are independent):

$$\begin{aligned} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\widetilde{\phi}(\mathbf{k}) \to a(\mathbf{k}), \qquad \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\widetilde{\phi}(-\mathbf{k}) \to b^\dagger(\mathbf{k}) \\ \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\widetilde{\phi}^*(\mathbf{k}) \to a^\dagger(\mathbf{k}), \qquad \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\widetilde{\phi}^*(-\mathbf{k}) \to b(\mathbf{k}). \end{aligned} \tag{5.37}$$

Since the classical field is now complex, the quantum field is not Hermitian:

$$\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\Big[a(\mathbf{k})e^{-i(\omega_{\mathbf{k}}x^0-\mathbf{k}\cdot\mathbf{x})} + b^\dagger(\mathbf{k})e^{i(\omega_{\mathbf{k}}x^0-\mathbf{k}\cdot\mathbf{x})}\Big] \\ \phi^\dagger(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\Big[b(\mathbf{k})e^{-i(\omega_{\mathbf{k}}x^0-\mathbf{k}\cdot\mathbf{x})} + a^\dagger(\mathbf{k})e^{i(\omega_{\mathbf{k}}x^0-\mathbf{k}\cdot\mathbf{x})}\Big]. \end{aligned} \tag{5.38}$$

For complex scalar fields, the commutation relations for the creation and annihilation operators (now there are two sets) are

$$\big[a(\mathbf{k}), a^\dagger(\mathbf{k}')\big] = \big[b(\mathbf{k}), b^\dagger(\mathbf{k}')\big] = \delta^3(\mathbf{k}-\mathbf{k}'). \tag{5.39}$$

All the other commutators are equal to zero.

What do the two sets of creation and annihilation operators correspond to? We can work out the renormalized charge operator:

$$\begin{aligned} Q_{\text{ren}} &= e\int d^3k\, \mathrm{N}\big[a^\dagger(\mathbf{k})a(\mathbf{k}) - b(\mathbf{k})b^\dagger(\mathbf{k})\big] \\ &= e\int d^3k\, \big[a^\dagger(\mathbf{k})a(\mathbf{k}) - b^\dagger(\mathbf{k})b(\mathbf{k})\big]. \end{aligned} \tag{5.40}$$

On the other hand, the renormalized Hamiltonian is

$$H_{\text{ren}} = \int d^3k\, \omega_{\mathbf{k}}\big[a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})\big]. \tag{5.41}$$

Hence, $a^\dagger(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ can be interpreted as creation operators for particles and antiparticles, respectively. Thus, the quantum theory of complex scalar fields predicts the existence of antiparticles.

6 Canonical Quantization of Electromagnetic Fields

6.1 Quantization in the Coulomb gauge

Coulomb gauge conditions:

$$A_0 = 0, \qquad \partial_i A^i = 0. \tag{6.1}$$

Then the Lagrangian of the electromagnetic field (see Eq. (4.86)) in the Coulomb gauge is

$$\begin{aligned}\mathcal{L}_{\rm CG} \equiv &-\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)\\ =&-\frac{1}{2}\big(\partial_\mu A_i \partial^\mu A^i - \partial_i A_j \partial^j A^i\big) \qquad (A^0 = 0)\\ =&-\frac{1}{2}\big(\partial_0 A_i \partial^0 A^i + \partial_j A_i \partial^j A^i - \cancel{\partial^j A_j \partial_i A^i}\big)\\ =&-\frac{1}{2}\big(\partial_0 A_i \partial^0 A^i + \partial_j A_i \partial^j A^i\big).\end{aligned} \tag{6.2}$$

Then the momentum conjugate to A_i is

$$\begin{aligned}\Pi^i :=& \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)}\\ =& -\partial^0 A^i = -F^{0i} = E^i,\end{aligned} \tag{6.3}$$

and the Hamiltonian density is

$$\begin{aligned}\mathcal{H} :=& \Pi^i \partial_0 A_i - \mathcal{L}\\ =& -\frac{1}{2}\partial_0 A_i \partial^0 A^i + \frac{1}{2}\partial_j A_i \partial^j A^i\\ =& \frac{1}{2}(\partial_0 A_i \partial_0 A_i + \partial_j A_i \partial_j A_i).\end{aligned} \tag{6.4}$$

The canonical commutation relations in the Coulomb guage are somewhat unusual (to be consistent with the Coulomb gauge condition):

$$\big[A^i(x^0,\mathbf{x}), \Pi^j(x^0,\mathbf{x}')\big] = i\int \frac{d^3k}{(2\pi)^3}\left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}\right)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = i\left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right)\delta^3(\mathbf{x}-\mathbf{x}'). \tag{6.5}$$

As usual, we define the creation and annihilation operators by replacing the Fourier images in the classical solution Eq. (4.76) by:

$$\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\widetilde{A}_\lambda(\mathbf{k}) \to a_\lambda(\mathbf{k}), \qquad \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\widetilde{A}^*_\lambda(\mathbf{k}) \to a^\dagger_\lambda(\mathbf{k}). \tag{6.6}$$

Thus the quantum EM field in the Coulomb gauge is

$$A^i(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\sum_{\lambda=1}^{2}\epsilon^i_\lambda(\mathbf{k})\Big[a_\lambda(\mathbf{k})e^{-i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} + a^\dagger_\lambda(\mathbf{k})e^{i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})}\Big] \tag{6.7}$$

To obtain the commutation relations for the creation and annihilation operators, we need to use the commutation relation (6.5). First, we write down the conjugate momentum as

$$\Pi^i(x) = -\partial^0 A^i = \int \frac{d^3k\,(i\omega_{\mathbf{k}})}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\sum_{\lambda=1}^{2}\epsilon^i_\lambda(\mathbf{k})\Big[-a_\lambda(\mathbf{k})e^{-i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} + a^\dagger_\lambda(\mathbf{k})e^{i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})}\Big]. \tag{6.8}$$

Similarly to Eq. (5.10), we take the partial Fourier transform of $A^i(x)$ and $\Pi^i(x)$:

$$\begin{aligned} A^i(x^0, \mathbf{k}) &= \int \frac{d^3x}{(2\pi)^3} A^i(x) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ \Pi^i(x^0, \mathbf{k}) &= \int \frac{d^3x}{(2\pi)^3} \Pi^i(x) e^{-i\mathbf{k}\cdot\mathbf{x}}. \end{aligned} \tag{6.9}$$

Then we have

$$\begin{aligned} A^i(x^0, \mathbf{k}) &= \int \frac{d^3x\, d^3k'}{(2\pi)^3 (2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}'}}} \sum_{\lambda=1}^{2} \epsilon^i_\lambda(\mathbf{k}') \left[a_\lambda(\mathbf{k}') e^{-i\omega_{\mathbf{k}'}x^0} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} + a^\dagger_\lambda(\mathbf{k}') e^{i\omega_{\mathbf{k}'}x^0} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \right] \\ &= \int \frac{d^3x\, d^3k'}{(2\pi)^3 (2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}'}}} \sum_{\lambda=1}^{2} \left[\epsilon^i_\lambda(\mathbf{k}') a_\lambda(\mathbf{k}') e^{-i\omega_{\mathbf{k}'}x^0} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} + \epsilon^i_\lambda(-\mathbf{k}') a^\dagger_\lambda(-\mathbf{k}') e^{i\omega_{\mathbf{k}'}x^0} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \right] \\ &\quad \left(\text{Note: } \delta^3(\mathbf{k}-\mathbf{k}') = \int \frac{d^3x}{(2\pi)^3} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \right) \\ &= \frac{1}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \sum_{\lambda=1}^{2} \left[\epsilon^i_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{-i\omega_{\mathbf{k}}x^0} + \epsilon^i_\lambda(-\mathbf{k}) a^\dagger_\lambda(-\mathbf{k}) e^{i\omega_{\mathbf{k}}x^0} \right]. \end{aligned} \tag{6.10}$$

Similarly,

$$\Pi^i(x^0, \mathbf{k}) = \frac{i\omega_{\mathbf{k}}}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \sum_{\lambda=1}^{2} \left[-\epsilon^i_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{-i\omega_{\mathbf{k}}x^0} + \epsilon^i_\lambda(-\mathbf{k}) a^\dagger_\lambda(-\mathbf{k}) e^{i\omega_{\mathbf{k}}x^0} \right]. \tag{6.11}$$

We adopt the following normalization conditions for the polarization vectors

$$\begin{aligned} &\epsilon^i_\lambda(\mathbf{k})\epsilon^i_{\lambda'}(\mathbf{k}) = \delta_{\lambda\lambda'}, \\ &\epsilon^i_\lambda(-\mathbf{k})\epsilon^i_{\lambda'}(-\mathbf{k}) = \delta_{\lambda\lambda'}, \\ &\epsilon^i_\lambda(\mathbf{k})\epsilon^i_{\lambda'}(-\mathbf{k}) = (-1)^\lambda \delta_{\lambda\lambda'}. \end{aligned} \tag{6.12}$$

Multiplying both sides of Eqs. (6.10) and (6.11) by $\epsilon^i_\lambda(\mathbf{k})$, we have

$$\begin{aligned} \epsilon^i_\lambda(\mathbf{k}) A^i(x^0, \mathbf{k}) &= \frac{1}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \left[a_\lambda(\mathbf{k}) e^{-i\omega_{\mathbf{k}}x^0} + (-1)^\lambda a^\dagger_\lambda(-\mathbf{k}) e^{i\omega_{\mathbf{k}}x^0} \right] \\ \epsilon^i_\lambda(\mathbf{k}) \Pi^i(x^0, \mathbf{k}) &= \frac{i\omega_{\mathbf{k}}}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \left[-a_\lambda(\mathbf{k}) e^{-i\omega_{\mathbf{k}}x^0} + (-1)^\lambda a^\dagger_\lambda(-\mathbf{k}) e^{i\omega_{\mathbf{k}}x^0} \right]. \end{aligned} \tag{6.13}$$

Then

$$\begin{aligned} a_\lambda(\mathbf{k}) &= \frac{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}{2} \epsilon^i_\lambda(\mathbf{k}) \left[A^i(x^0, \mathbf{k}) + \frac{i}{\omega_{\mathbf{k}}} \Pi^i(x^0, \mathbf{k}) \right] e^{i\omega_{\mathbf{k}}x^0} \\ &= \int \frac{d^3x}{(2\pi)^3} \frac{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}{2} \epsilon^i_\lambda(\mathbf{k}) \left[A^i(x) + \frac{i}{\omega_{\mathbf{k}}} \Pi^i(x) \right] e^{i\omega_{\mathbf{k}}x^0 - i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3x}{(2\pi)^{3/2}} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \epsilon^i_\lambda(\mathbf{k}) \left[A^i(x) + \frac{i}{\omega_{\mathbf{k}}} \Pi^i(x) \right] e^{i\omega_{\mathbf{k}}x^0 - i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{6.14}$$

where in the second line, we have used Eq. (6.9).
Similarly, to obtain an expression for $a^\dagger_\lambda(\mathbf{k})$, we multiply both sides of Eqs. (6.10) and (6.11) by $\epsilon^i_\lambda(-\mathbf{k})$,

$$\begin{aligned} \epsilon^i_\lambda(-\mathbf{k}) A^i(x^0, \mathbf{k}) &= \frac{1}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \left[(-1)^\lambda a_\lambda(\mathbf{k}) e^{-i\omega_{\mathbf{k}}x^0} + a^\dagger_\lambda(-\mathbf{k}) e^{i\omega_{\mathbf{k}}x^0} \right] \\ \epsilon^i_\lambda(-\mathbf{k}) \Pi^i(x^0, \mathbf{k}) &= \frac{i\omega_{\mathbf{k}}}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \left[-(-1)^\lambda a_\lambda(\mathbf{k}) e^{-i\omega_{\mathbf{k}}x^0} + a^\dagger_\lambda(-\mathbf{k}) e^{i\omega_{\mathbf{k}}x^0} \right]. \end{aligned} \tag{6.15}$$

Therefore,

$$
\begin{aligned}
a^\dagger_\lambda(-\mathbf{k}) &= \frac{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{k}}}{2}\epsilon^i_\lambda(-\mathbf{k})\Big[A^i(x^0,\mathbf{k}) - \frac{i}{\omega_\mathbf{k}}\Pi^i(x^0,\mathbf{k})\Big]e^{-i\omega_\mathbf{k}x^0} \\
&= \int \frac{d^3x}{(2\pi)^{3/2}}\sqrt{\frac{\omega_\mathbf{k}}{2}}\epsilon^i_\lambda(-\mathbf{k})\Big[A^i(x) - \frac{i}{\omega_\mathbf{k}}\Pi^i(x)\Big]e^{-i\omega_\mathbf{k}x^0 - i\mathbf{k}\cdot\mathbf{x}} \\
\implies \quad a^\dagger_\lambda(\mathbf{k}) &= \int \frac{d^3x}{(2\pi)^{3/2}}\sqrt{\frac{\omega_\mathbf{k}}{2}}\epsilon^i_\lambda(\mathbf{k})\Big[A^i(x) - \frac{i}{\omega_\mathbf{k}}\Pi^i(x)\Big]e^{-i\omega_\mathbf{k}x^0 + i\mathbf{k}\cdot\mathbf{x}}.
\end{aligned}
\tag{6.16}
$$

Now we can construct the commutator for $a_\lambda(\mathbf{k})$ and $a^\dagger_{\lambda'}(\mathbf{k}')$:

$$
\begin{aligned}
\Big[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')\Big] &= \int \frac{d^3x\, d^3x'}{(2\pi)^3}\frac{\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}}{2}\,\epsilon^i_\lambda(\mathbf{k})\epsilon^j_{\lambda'}(\mathbf{k}')\Big[A^i(x^0,\mathbf{x}) + \frac{i}{\omega_\mathbf{k}}\Pi^i(x^0,\mathbf{x}), A^j(x^0,\mathbf{x}') - \frac{i}{\omega_{\mathbf{k}'}}\Pi^j(x^0,\mathbf{x}')\Big] \\
&\quad e^{-i(\omega_{\mathbf{k}'}-\omega_\mathbf{k})x^0}e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')} \\
&= \int \frac{d^3x\, d^3x'}{(2\pi)^3}\frac{\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}}{2}\,\frac{\epsilon^i_\lambda(\mathbf{k})\epsilon^j_{\lambda'}(\mathbf{k}')}{i}\Bigg\{\frac{1}{\omega_{\mathbf{k}'}}\underbrace{\Big[A^i(x^0,\mathbf{x}),\Pi^j(x^0,\mathbf{x}')\Big]}_{=i(\delta^{ij}-\partial^i\partial^j/\nabla^2)\delta^3(\mathbf{x}-\mathbf{x}')} + \frac{1}{\omega_\mathbf{k}}\underbrace{\Big[A^j(x^0,\mathbf{x}'),\Pi^i(x^0,\mathbf{x})\Big]}_{=i(\delta^{ji}-\partial^j\partial^i/\nabla^2)\delta^3(\mathbf{x}'-\mathbf{x})}\Bigg\} \\
&\quad e^{-i(\omega_{\mathbf{k}'}-\omega_\mathbf{k})x^0}e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')} \\
&= \int \frac{d^3x}{(2\pi)^3}\frac{\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}}{2}\bigg(\frac{1}{\omega_{\mathbf{k}'}} + \frac{1}{\omega_\mathbf{k}}\bigg)\epsilon^i_\lambda(\mathbf{k})\epsilon^j_{\lambda'}(\mathbf{k}')\bigg(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\bigg)e^{-i(\omega_{\mathbf{k}'}-\omega_\mathbf{k})x^0}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}.
\end{aligned}
\tag{6.17}
$$

Notice that

$$
\begin{aligned}
&\partial^i\partial^j\Big[e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\Big]\epsilon^i_\lambda(\mathbf{k})\epsilon^j_{\lambda'}(\mathbf{k}') \\
&= -\,(k_i - k'_i)\epsilon^i_\lambda(\mathbf{k})\big(k_j - k'_j\big)\epsilon^j_{\lambda'}(\mathbf{k}')e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\
&= -\,k'_i\epsilon^i_\lambda(\mathbf{k})\;k'_j\epsilon^j_{\lambda'}(\mathbf{k}')e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}, \qquad (\text{since } k_i\epsilon^i_\lambda(\mathbf{k}) = 0)
\end{aligned}
\tag{6.18}
$$

and then

$$
\underbrace{\int \frac{d^3x}{(2\pi)^3}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}}_{=\delta^3(\mathbf{k}-\mathbf{k}')}\;\Big(k'_i\epsilon^i_\lambda(\mathbf{k})\Big)\Big(k'_j\epsilon^j_{\lambda'}(\mathbf{k}')\Big) = 0 \tag{6.19}
$$

because if $\mathbf{k} \neq \mathbf{k}'$, $\delta^3(\mathbf{k}-\mathbf{k}') = 0$ and the integral is zero. But if $\mathbf{k} = \mathbf{k}'$, then $k'_i\epsilon^i_\lambda(\mathbf{k}) = k_i\epsilon^i_\lambda(\mathbf{k}) = 0$ and $k'_j\epsilon^j_{\lambda'}(\mathbf{k}') = k_j\epsilon^j_{\lambda'}(\mathbf{k}') = 0$. Thus the integral is zero for any $\mathbf{k}$ and $\mathbf{k}'$.
So Eq. (6.17) becomes

$$
\begin{aligned}
\Big[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')\Big] &= \frac{\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}}{2}\bigg(\frac{1}{\omega_{\mathbf{k}'}} + \frac{1}{\omega_\mathbf{k}}\bigg)e^{-i(\omega_{\mathbf{k}'}-\omega_\mathbf{k})x^0}\,\epsilon^i_\lambda(\mathbf{k})\epsilon^i_{\lambda'}(\mathbf{k}')\delta^3(\mathbf{k}-\mathbf{k}') \\
&= \frac{\omega_\mathbf{k}}{2}\frac{2}{\omega_\mathbf{k}}\underbrace{e^{-i(\omega_\mathbf{k}-\omega_\mathbf{k})x^0}}_{=e^0=1}\underbrace{\epsilon^i_\lambda(\mathbf{k})\epsilon^i_{\lambda'}(\mathbf{k})}_{=\delta_{\lambda\lambda'}}\delta^3(\mathbf{k}-\mathbf{k}') \qquad (\mathbf{k} = \mathbf{k}' \text{ is non-vanishing}) \\
&= \delta_{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}').
\end{aligned}
\tag{6.20}
$$

Therefore, we obtain the commutation relation for the creation and annihilation operators in the Coulomb gauge:

$$
\Big[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')\Big] = \delta_{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}'). \tag{6.21}
$$

Other commutators give zeroes.
The renormalized Hamiltonian is

$$
H_{\text{ren}} = \int d^3k\;\omega_\mathbf{k}\sum_{\lambda=1}^{2} a^\dagger_\lambda(\mathbf{k})a_\lambda(\mathbf{k}). \tag{6.22}
$$

6.2 Quantization in the Lorenz gauge

According to the standard definition, the conjugate momentum of the vector field $A_\mu(x)$ is

$$\Pi^\mu(x) := \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu}. \tag{6.23}$$

This means that A_0 component does not have a canonical momentum, as $\Pi^0 = -F^{00}$ is identically zero. However, in the covariant quantization all four components of A_μ and Π^μ should be brought into play. To achieve this, we will need to change the initial Lagrangian for the EM field to find Π^0 that does not vanish. The only reasonable change is such that it incorporates the covariant gauge condition $\partial_\mu A^\mu$, and hence gives the equation of motion in the Lorenz gauge (4.80).The Lagrangian that does the job is

$$\mathcal{L}_{\text{LG}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu A^\mu \partial_\nu A^\nu, \tag{6.24}$$

where the 2nd term (extra term) is called the *gauge-fixing term*. This gauge is also called the **Feynman gauge**. From the Lagrangian we calculate the canonical conjugate momentum to be

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} - \eta^{\mu 0}(\partial_\mu A^\mu). \tag{6.25}$$

So we have

$$\Pi^0 = -\partial_\mu A^\mu, \tag{6.26}$$

and $\Pi^i = -F^{0i}$ as before.
Furthermore, we can use integration by parts to rewrite the Lorenz gauge Lagrangian as

$$\begin{aligned} \mathcal{L}_{\text{LG}} &= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2}\partial_\mu A^\mu \partial_\nu A^\nu \\ &= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu [A_\nu(\partial^\nu A^\mu) - (\partial_\nu A^\nu)A^\mu] \end{aligned} \tag{6.27}$$

so that the last term is a four-divergence which has no influence on the field equations. Thus the dynamics of the EM field in the Lorenz gauge can be described by the simple Lagrangian

$$\mathcal{L}'_{\text{LG}} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu. \tag{6.28}$$

Then the canonical conjugate momentum is just given by

$$\Pi^\mu = \frac{\partial \mathcal{L}'_{\text{LG}}}{\partial(\partial_0 A_\mu)} = -\partial^0 A^\mu. \tag{6.29}$$

Now it is time to impose the Lorenz gauge condition. Unfortunately, the old condition $\partial_\mu A^\mu$ does not work because it would make the conjugate momentum in Eq. (6.26) vanish again. So we impose a weaker condition that, for any *physical state* $|\psi\rangle$, the expectation value of the operator $\partial_\mu A^\mu$ is zero, i.e.

$$\langle\psi|\partial_\mu A^\mu|\psi\rangle = 0. \tag{6.30}$$

This is also called the *Gupta-Bleuler quantization condition*. More precisely, given the Fock space $\mathcal{F}$, we define the subset $\mathcal{F}'$ of physical states as the states $|\psi\rangle$ satisfying Eq. (6.30). We stress that the above condition is not a constraint on the field A_μ, but a restriction of the states of $\mathcal{F}$: only a subset of them (physical states) is selected. Similar to Eq. (6.7), the field expansion is written as

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \sum_{\lambda=0}^{3} \epsilon_\lambda^\mu(\mathbf{k}) \Big[a_\lambda(\mathbf{k})e^{-i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} + a_\lambda^\dagger(\mathbf{k})e^{i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})}\Big]. \tag{6.31}$$

We introduce the equal-time commutation relations in the Lorenz gauge:

$$\begin{aligned} \left[A^{\mu}(x^0,\mathbf{x}),\Pi^{\nu}(x^0,\mathbf{x}')\right] &= i\eta^{\mu\nu}\delta^3(\mathbf{x}-\mathbf{x}') \\ \left[A^{\mu}(x^0,\mathbf{x}),A^{\nu}(x^0,\mathbf{x}')\right] &= \left[\Pi^{\mu}(x^0,\mathbf{x}),\Pi^{\nu}(x^0,\mathbf{x}')\right] = 0. \end{aligned} \tag{6.32}$$

Using Eq. (6.29) as the conjugate momentum and following the same procedure as in the case of Coulomb gauge, we can obtain the commutation relations for the creation and annihilation operators in the Lorenz gauge:

$$\begin{aligned} \left[a_{\lambda}(\mathbf{k}),a^{\dagger}_{\lambda'}(\mathbf{k}')\right] &= -\eta_{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}') \\ \left[a_{\lambda}(\mathbf{k}),a_{\lambda'}(\mathbf{k}')\right] &= \left[a^{\dagger}_{\lambda}(\mathbf{k}),a^{\dagger}_{\lambda'}(\mathbf{k}')\right] = 0. \end{aligned} \tag{6.33}$$

We can decompose the field operator $A^{\mu}(x)$ into positive (A^{μ}_{+}) and negative (A^{μ}_{-}) frequency modes:

$$\begin{aligned} A^{\mu}_{+}(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \sum_{\lambda=0}^{3} \epsilon^{\mu}_{\lambda}(\mathbf{k})\left[a_{\lambda}(\mathbf{k})e^{-i(\omega_{\mathbf{k}}x^0-\mathbf{k}\cdot\mathbf{x})}\right] \\ A^{\mu}_{-}(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \sum_{\lambda=0}^{3} \epsilon^{\mu}_{\lambda}(\mathbf{k})\left[a^{\dagger}_{\lambda}(\mathbf{k})e^{i(\omega_{\mathbf{k}}x^0-\mathbf{k}\cdot\mathbf{x})}\right]. \end{aligned} \tag{6.34}$$

Then the gauge condition implies

$$\begin{aligned} & \langle\psi|\partial_{\mu}A^{\mu}|\psi\rangle = 0 = \langle\psi|\big(\partial_{\mu}A^{\mu}_{+}+\partial_{\mu}A^{\mu}_{-}\big)|\psi\rangle \\ \Longrightarrow\quad & \langle\psi|\partial_{\mu}A^{\mu}_{+}|\psi\rangle = -\langle\psi|\partial_{\mu}A^{\mu}_{-}|\psi\rangle = \big(\langle\psi|\partial_{\mu}A^{\mu}_{+}|\psi\rangle\big)^{*} \\ \Longrightarrow\quad & \partial_{\mu}A^{\mu}_{+}\,|\psi\rangle = 0. \end{aligned} \tag{6.35}$$

In turn, this condition implies

$$\sum_{\lambda=0}^{3}\left[k_i\epsilon^{i}_{\lambda}(\mathbf{k})+\omega_{\mathbf{k}}\epsilon^{0}_{\lambda}(\mathbf{k})\right]a_{\lambda}(\mathbf{k})\,|\psi\rangle = 0, \tag{6.36}$$

which, in view of $k_{\mu}\epsilon^{\mu}_{1,2}=0$, gives

$$\left[\big(k_i\epsilon^{i}_{0}(\mathbf{k})+\omega_{\mathbf{k}}\epsilon^{0}_{0}(\mathbf{k})\big)a_0(\mathbf{k})+\big(k_i\epsilon^{i}_{3}(\mathbf{k})+\omega_{\mathbf{k}}\epsilon^{0}_{3}(\mathbf{k})\big)a_3(\mathbf{k})\right]|\psi\rangle = 0. \tag{6.37}$$

Finally, combining $k_{\mu}\epsilon^{\mu}_{1,2}=0$ with Eq. (4.83), we have $k_{\mu}\epsilon^{\mu}_{0}+k_{\mu}\epsilon^{\mu}_{3}=0$, i.e.

$$\big(k_i\epsilon^{i}_{0}(\mathbf{k})+\omega_{\mathbf{k}}\epsilon^{0}_{0}(\mathbf{k})\big) = -\big(k_i\epsilon^{i}_{3}(\mathbf{k})+\omega_{\mathbf{k}}\epsilon^{0}_{3}(\mathbf{k})\big). \tag{6.38}$$

Then we obtain

$$\left[a_0(\mathbf{k})-a_3(\mathbf{k})\right]|\psi\rangle = 0. \tag{6.39}$$

Hence,

$$\langle\psi|\left[a^{\dagger}_{0}(\mathbf{k})a_0(\mathbf{k})-a^{\dagger}_{3}(\mathbf{k})a_3(\mathbf{k})\right]|\psi\rangle = 0, \tag{6.40}$$

suggesting that the contributions to the quantum Hamiltonian of the EM field from the scalar and longitudinal photons are canceled out, leaving only the contributions from the transverse (physical) photons.

7 Canonical Quantization of Dirac Fields

7.1 Quantization with anti-commutation relations

The first step to quantize the Dirac field is the familiar procedure of replacing the Fourier images of Eqs. (4.47) and (4.48) by the creation and annihilation operators:

$$\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}c_s(\mathbf{k}) \to c_s(\mathbf{k}), \qquad \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}d_s(-\mathbf{k}) \to d_s^\dagger(\mathbf{k}). \tag{7.1}$$

Since the Dirac field is complex, we have two sets of creation and annihilation operators. Then the positive and negative frequency modes can be written as

$$\begin{aligned} \psi_+(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \sum_s c_s(\mathbf{k}) u_s(\mathbf{k}) e^{-i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} \\ \psi_-(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \sum_s d_s^\dagger(\mathbf{k}) v_s(-\mathbf{k}) e^{i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})}. \end{aligned} \tag{7.2}$$

Therefore, the full quantum Dirac field is the sum of them

$$\psi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \sum_s \Big[c_s(\mathbf{k}) u_s(\mathbf{k}) e^{-i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} + d_s^\dagger(\mathbf{k}) v_s(-\mathbf{k}) e^{i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} \Big]. \tag{7.3}$$

Similarly, the Dirac adjoint of the field is

$$\begin{aligned} \overline{\psi}(x) &= \overline{\psi}_+(x) + \overline{\psi}_-(x) \\ &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} \sum_s \Big[d_s(\mathbf{k}) \overline{v}_s(-\mathbf{k}) e^{-i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} + c_s^\dagger(\mathbf{k}) \overline{u}_s(\mathbf{k}) e^{i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})} \Big]. \end{aligned} \tag{7.4}$$

The conjugate momentum to the Dirac field operator is also straightforward to find:

$$\Pi(x) := \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)} = i\overline{\psi}(x)\gamma^0 = i\psi^\dagger(x). \tag{7.5}$$

Here comes the key difference from the case of quantizing bosonic fields: instead of the commutation relations, we need to impose the **anti-commutation relations** to the Dirac fields:

$$\begin{aligned} &\left\{\psi(x^0,\mathbf{x}), \psi^\dagger(x^0,\mathbf{x}')\right\} = \delta^3(\mathbf{x}-\mathbf{x}') \\ &\left\{\psi(x^0,\mathbf{x}), \psi(x^0,\mathbf{x}')\right\} = \left\{\psi^\dagger(x^0,\mathbf{x}), \psi^\dagger(x^0,\mathbf{x}')\right\} = 0. \end{aligned} \tag{7.6}$$

Then the anti-commutation relations among the creation and annihilation operators follow

$$\begin{aligned} &\left\{c_s(\mathbf{k}), c_{s'}^\dagger(\mathbf{k}')\right\} = \delta_{ss'}\delta^3(\mathbf{k}-\mathbf{k}') \\ &\left\{d_s(\mathbf{k}), d_{s'}^\dagger(\mathbf{k}')\right\} = \delta_{ss'}\delta^3(\mathbf{k}-\mathbf{k}'), \end{aligned} \tag{7.7}$$

and all the other anti-commutators are equal to zero.

The effects of anti-commutation relations we have imposed above can be seen immediately. First recall that, when discussing the classical Dirac field, we encountered a problem where the negative frequency modes had negative contribution to the Hamiltonian so that the total Hamiltonian is not positive definite. The anti-commutation relations here solve this problem. Due to the anti-commutation relations, each change in the order of two spinor operators under the operation of normal ordering gives an extra minus sign compared to the case of bosonic operators. Therefore, the normal-ordered (physical) Hamiltonian now becomes (refer to Eq. (4.52))

$$\begin{aligned} H_{\text{ren}} &= \int d^3k\, \omega_{\mathbf{k}} \sum_s \mathrm{N}\big[c_s^\dagger(\mathbf{k})c_s(\mathbf{k}) - d_s(\mathbf{k})d_s^\dagger(\mathbf{k})\big] \\ &= \int d^3k\, \omega_{\mathbf{k}} \sum_s \big[c_s^\dagger(\mathbf{k})c_s(\mathbf{k}) + d_s^\dagger(\mathbf{k})d_s(\mathbf{k})\big]. \end{aligned} \tag{7.8}$$

Thus the anti-commutation relations indeed ensure that the physical Hamiltonian is positive definite.
We can also calculate the renormalized (normal-ordered) charge operator from the corresponding classical expression Eq. (4.57):

$$\begin{aligned} Q_{\text{ren}} &= e\int d^3k\, \sum_s \mathrm{N}\big[c_s^\dagger(\mathbf{k})c_s(\mathbf{k}) + d_s(\mathbf{k})d_s^\dagger(\mathbf{k})\big] \\ &= e\int d^3k\, \sum_s \big[c_s^\dagger(\mathbf{k})c_s(\mathbf{k}) - d_s^\dagger(\mathbf{k})d_s(\mathbf{k})\big]. \end{aligned} \tag{7.9}$$

7.2 Spin-statistics relation and probabilities in QFT

Consider a state of two bosonic particles with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$:

$$\begin{aligned} |\mathbf{k}_1, \mathbf{k}_2\rangle &= a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\,|0,0\rangle \\ &= a^\dagger(\mathbf{k}_2)a^\dagger(\mathbf{k}_1)\,|0,0\rangle \\ &= |\mathbf{k}_2, \mathbf{k}_1\rangle\,. \end{aligned} \tag{7.10}$$

This shows that the quantum state is symmetric under the exchange of bosonic particles. This symmetry is known as the *Bose symmetry*, which stems from the canonical commutation relations of the bosonic field operators.
Now consider two Dirac particles with momenta $\mathbf{k}_1$, $\mathbf{k}_2$ and spins s_1, s_2, respectively. The two-particle state is labelled as $|\mathbf{k}_1, s_1; \mathbf{k}_2, s_2\rangle$. We have

$$\begin{aligned} |\mathbf{k}_1, s_1; \mathbf{k}_2, s_2\rangle &= c_{s_1}^\dagger c_{s_2}^\dagger\,|0,0\rangle \\ &= -c_{s_2}^\dagger c_{s_1}^\dagger\,|0,0\rangle \\ &= -\,|\mathbf{k}_2, s_2; \mathbf{k}_1, s_1\rangle\,. \end{aligned} \tag{7.11}$$

Thus, under the exchange of two Dirac particles, the two-particle state changes the sign. This is the so-called *Fermi-Dirac symmetry*, and the particles satisfying this property are called fermions. The *Pauli exclusion principle* immediately follows from Eq. (7.11), which is in turn resulted from the anti-commutation relations. Indeed, the state with two (or more) fermions with the same mass and momentum $\mathbf{k}_1 = \mathbf{k}_2$ and the same spin polarizations $s_1 = s_2$ does not exist because $|\mathbf{k}, s; \mathbf{k}, s\rangle = 0$.
Conclusion: The consistent quantization requires to use commutation relations for the bosonic fields and anti-commutation relations for the fermionic fields. This is known as the **spin-statistics relation**.

In relativistic quantum mechanics, there is a problem with the relativistic probability: it is not positive definite for the complex fields and trivially vanishes for the real fields. The reason is that we treated the fields as wave functions of single-particle states in quantum mechanics. However, in QFT, the fields are operators and the wave functions are defined as expectation values of these operators. Consider a real scalar field, we can calculate the wave function corresponding to a single-particle state:

$$\begin{aligned} \Phi(x) &= \langle 0|\phi(x)|\mathbf{k}\rangle \\ &= \int \frac{d^3k'}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}'}}}\Big[\langle 0|a(\mathbf{k}')|\mathbf{k}\rangle\, e^{-i(\omega_{\mathbf{k}'}x^0 - \mathbf{k}'\cdot\mathbf{x})} + \langle 0|a^\dagger(\mathbf{k}')|\mathbf{k}\rangle\, e^{i(\omega_{\mathbf{k}'}x^0 - \mathbf{k}'\cdot\mathbf{x})}\Big] \\ &= \int \frac{d^3k'}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}'}}} \underbrace{\langle \mathbf{k}'|\mathbf{k}\rangle}_{=\delta^3(\mathbf{k}-\mathbf{k}')}\; e^{-i(\omega_{\mathbf{k}'}x^0 - \mathbf{k}'\cdot\mathbf{x})} \\ &= \frac{d^3k'}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}} e^{-i(\omega_{\mathbf{k}}x^0 - \mathbf{k}\cdot\mathbf{x})}, \end{aligned} \tag{7.12}$$

where in the third line we have used $\langle 0| a(\mathbf{k}') = \left(a^\dagger(\mathbf{k}') |0\rangle\right)^* = \langle \mathbf{k}'|$ and $\langle 0| a^\dagger(\mathbf{k}') = (a(\mathbf{k}') |0\rangle)^* = 0$.
Now the relativistic probability density is

$$\rho = i\big(\Phi^* \partial^0 \Phi - \Phi \partial^0 \Phi^*\big). \tag{7.13}$$

Using Eq. (7.12) we find

$$\rho = \frac{1}{(2\pi)^3}, \tag{7.14}$$

which is non-zero and positive! This is the probability to find a spinless particle within a unit volume.

8 Interaction Fields

8.1 Some examples of interacting theories

So far we have dealt with free field theories, which we are able to solve exactly. However, the dynamics of a free field theory is rather trivial: one decides which state (or superposition of states) the field is in at some initial time, and the field will remain in the same state for all subsequent times. Moreover, free fields are metaphysical objects as they are not detected in experiments. Therefore, we are more interested in more realistic situations where the fields interact with each other.

Self-interacting scalar field

We denote the free Lagrangian of a scalar field given by Eq. (4.1) by $\mathcal{L}_0$. We can add an arbitrary function of the scalar field $V(\phi)$ to $\mathcal{L}_0$, which is automatically invariant under the Poincare transformations. The simplest possibility is the polynomial interactions:

$$V(\phi) = \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4 + \cdots + \frac{\lambda_n}{n!}\phi^n. \tag{8.1}$$

The parameters λ_k, known as the *coupling constants*, dictates the strength of the corresponding k-scalar interactions. Note that the coupling constants λ_k with $k > 4$ have negative mass dimensions. If we assume that $\lambda_k \sim M^{4-k}$, where M is the typical mass scale of a theory, and $|\phi| \ll M$, then all the terms with $k > 4$ can be ignored compared with the cubic and quartic interactions because they are suppressed by the heavy mass scale M. It turns out that upon quantization, only the cubic and quartic interactions give a theory which can be extrapolated to an arbitrary high energy scale.
Let us consider the ϕ^4-theory, i.e. $V(\phi) = \frac{\lambda}{4!}\phi^4$, where $\lambda := \lambda_4$. Then the Lagrangian is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{8.2}$$

Using the EL equation, we find that the equation of motion for this theory is

$$(\Box + m^2)\phi = -\frac{\lambda}{3!}\phi^3. \tag{8.3}$$

This equation, unlike the KG equation for a free scalar field, cannot be solved analytically!
We can also calculate the Hamiltonian density for the ϕ^4-theory:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \tag{8.4}$$

where

$$\mathcal{H}_0 = \frac{1}{2}\left[(\partial_0\phi)^2 + (\partial_i\phi)^2 + m^2\phi^2\right] \tag{8.5}$$

and

$$\mathcal{H}_I = -\mathcal{L}_I = \frac{\lambda}{4!}\phi^4. \tag{8.6}$$

Yukawa interactions

The **Yukawa interactions** are the couplings between scalar fields and fermion fields. We can write down a simple, relativistically invariant interacting Lagrangian (a Lorentz scalar) between scalar and Dirac fields as:

$$\mathcal{L}_I = y\overline{\psi}\psi\phi, \tag{8.7}$$

where y is the coupling constant known as the *Yukawa coupling*.
We have the following interaction Hamiltonian density:

$$\mathcal{H}_I = -\mathcal{L}_I = -y\overline{\psi}\psi\phi. \tag{8.8}$$

If we take the pseudoscalar scalar field φ intead of the scalar field and assume that the spatial inversion is a symmetry of the theory, we should couple the pseudoscalar to the combination of fermion fields which are pseudoscalars as well. The simplest such combination is $\overline{\psi}\gamma^5\psi$, and thus the interaction Lagrangian has the form

$$\mathcal{L}_I = g\overline{\psi}\gamma^5\psi\varphi, \tag{8.9}$$

where g is the coupling constant. This Lagrangian describes the interaction between the particles known as *pions* (pseudoscalars) and *nucleons* (protons + neutrons, which are fermions).

Interactions of fermions with gauge fields

It appears that all elementary matter fields observed so far in experiments are spin-1/2 fermions, while the fields which mediate fundamental interactions (gauge bosons except gravitons) are spin-1 vectors. Moreover, an important experimental fact is that the interactions of spinor fields with vector fields are not arbitrary but respect symmetries, namely the gauge symmetries. Recall that the Lagrangian for free EM fields $A_\mu(x)$ is invariant under the gauge $U(1)$ transformations:

$$\delta A_\mu = \partial_\mu \alpha(x). \tag{8.10}$$

Also recall that the Lagrangian for free Dirac fields is invariant under the global $U(1)$ transformations. Now consider making these transformation local (i.e. position-dependent):

$$\psi(x) \to e^{-ie\alpha(x)}\psi(x), \qquad \overline{\psi}(x) \to e^{ie\alpha(x)}\overline{\psi}(x), \tag{8.11}$$

or in the infinitesimal form,

$$\delta\psi = -ie\alpha(x)\psi, \qquad \delta\overline{\psi} = ie\alpha(x)\psi, \tag{8.12}$$

and try to write down an interaction Lagrangian between Dirac fermions and EM field so that the full Lagrangian of the system is invariant under the gauge transformations Eq. (8.10) and (8.12). It turns out that the interaction Lagrangian we are looking for has the form

$$\mathcal{L}_I = -e\overline{\psi}\gamma^\mu\psi A_\mu. \tag{8.13}$$

Let us check.
Under the gauge transformations Eq. (8.10) and (8.12) the interaction Lagrangian transforms as

$$\begin{aligned}
\delta\mathcal{L}_I &= \delta\big(-e\overline{\psi}\gamma^\mu\psi A_\mu\big) \\
&= -e(\delta\overline{\psi})\gamma^\mu\psi A_\mu - e\overline{\psi}\gamma^\mu(\delta\psi)A_\mu - e\overline{\psi}\gamma^\mu\psi(\delta A_\mu) \\
&= \cancel{-e\big(ie\alpha(x)\overline{\psi}\big)\gamma^\mu\psi A_\mu} \cancel{-e\overline{\psi}\gamma^\mu(-ie\alpha(x)\psi)A_\mu} - e\overline{\psi}\gamma^\mu\psi(\partial_\mu\alpha(x)) \\
&= -e\overline{\psi}\gamma^\mu\psi(\partial_\mu\alpha(x)).
\end{aligned} \tag{8.14}$$

And the free Dirac Lagrangian transforms as

$$\begin{aligned}
\delta\mathcal{L}_D &= \delta(i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi) \\
&= i\big(\delta\overline{\psi}\big)\gamma^\mu\partial_\mu\psi + i\overline{\psi}\gamma^\mu\partial_\mu(\delta\psi) - m\big(\delta\overline{\psi}\big)\psi - m\overline{\psi}(\delta\psi) \\
&= \cancel{-e\alpha(x)\overline{\psi}\gamma^\mu\partial_\mu\psi} + e\overline{\psi}\gamma^\mu\psi(\partial_\mu\alpha(x)) \cancel{+e\overline{\psi}\gamma^\mu\alpha(x)\partial_\mu\psi} \cancel{-iem\alpha(x)\overline{\psi}\psi} \cancel{+iem\alpha(x)\overline{\psi}\psi} \\
&= +e\overline{\psi}\gamma^\mu\psi(\partial_\mu\alpha(x)) = -\delta\mathcal{L}_I.
\end{aligned} \tag{8.15}$$

Therefore, the variation of the free Dirac Lagrangian exactly cancels out the variation of the interaction Lagrangian under the gauge transformations. So if we write the full Lagrangian as

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_M + \mathcal{L}_I, \tag{8.16}$$

where $\mathcal{L}_D = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi$ and $\mathcal{L}_M = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ are the free Lagrangians for the massive Dirac field and the EM field, respectively, it will be invariant under the gauge transformations Eq. (8.10) and (8.12).
We can combine the interaction term $\mathcal{L}_I$ with the kinetic part of $\mathcal{L}_D$ by introducing the so-called **covariant deriva-**

tive:

$$D_\mu = \partial_\mu + ieA_\mu. \tag{8.17}$$

Hence, the formal rule to make a theory gauge invariant is to replace the ordinary derivatives by the covariant ones.

8.2 The interaction picture and the S-matrix

Usually, the goal of interaction theories is to calculate the quantum mechanical amplitude for some initial state to change into some final state. From this we can calculate the transition probability, which is eventually expressed as a cross-section. Unfortunately, it is almost always impossible to calculate these probabilities exactly, and one must use a perturbation expansion in some small number which parameterizes the strength of the interaction - a coupling constant. The perturbation theory that we are going to consider below is only applicable when this coupling constant is small. Moreover, as we will see in next chapter, these perturbation expansions can be written down in a graphical way with the so-called *Feynman diagrams*.

Let us first recall some results from ordinary time-dependent perturbation theory in quantum mechanics. For interaction fields, we tend to work with the **interaction picture**.
Consider the following Hamiltonian:

$$H(t) = H_0 + H_{\text{int}}(t), \tag{8.18}$$

where $H_0 = p^2/2m$ represents the kinetic energy and is independent of time, while $H_{\text{int}}(t)$ corresponds to the interaction potential that is dependent on time.
The time evolution operator $U(t, t_0)$ satisfies the following equation (as in the Schrodinger picture $U(t) = e^{-iHt/\hbar}$),

$$\frac{\partial U(t,t_0)}{\partial t} = (-iH/\hbar)U(t,t_0) = (-iH_0/\hbar)U(t,t_0) - (iH_{\text{int}}/\hbar)U(t,t_0). \tag{8.19}$$

We introduce an operator $U_I(t, t_0)$, which is defined as

$$U_I(t,t_0) = e^{iH_0(t-t_0)/\hbar}U(t,t_0). \tag{8.20}$$

Taking the time derivative gives

$$\begin{aligned}
\frac{\partial U_I(t,t_0)}{\partial t} &= \left(i\frac{H_0}{\hbar}\right)e^{iH_0(t-t_0)/\hbar}U(t,t_0) + e^{iH_0(t-t_0)/\hbar}\frac{\partial U(t,t_0)}{\partial t} \\
&= \left(-\frac{i}{\hbar}\right)e^{iH_0(t-t_0)/\hbar}H_{\text{int}}(t)e^{-iH_0(t-t_0)/\hbar}U_I(t,t_0).
\end{aligned} \tag{8.21}$$

Therefore, we can define

$$H_I(t) = e^{iH_0(t-t_0)/\hbar}H_{\text{int}}e^{-iH_0(t-t_0)/\hbar} \tag{8.22}$$

as the interaction Hamiltonian in the interaction picture. Eq. (8.21) then reads

$$\frac{\partial U_I(t,t_0)}{\partial t} = -\frac{i}{\hbar}H_I(t)U_I(t,t_0). \tag{8.23}$$

Since $U(t_0, t_0) = 1$, we must have $U_I(t_0, t_0) = 1$. Integrating both sides of the equation above gives

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' H_I(t')U_I(t',t_0). \tag{8.24}$$

Through the recursion Eq. (8.24), we get the following series expansion:

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' H_I(t') + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' H_I(t') \int_{t_0}^{t'} dt'' H_I(t'') + \cdots \tag{8.25}$$

By defining the *time-ordered product* of two operators $A(t')$ and $B(t'')$ as

$$\text{T}[A(t')B(t'')] = \theta(t'-t'')A(t')B(t'') + \theta(t''-t')B(t'')A(t'), \tag{8.26}$$

where the θ-function is the unit step function defined by

$$\theta(t_1 - t_2) = \begin{cases} 0, & t_1 < t_2; \\ 1, & t_1 > t_2, \end{cases} \tag{8.27}$$

we can write the integral in the second term of Eq. (8.25)

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'') = \frac{1}{2} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \mathrm{T}[H_I(t') H_I(t'')]. \tag{8.28}$$

Hence, the expansion can be written as

$$U_I(t, t_0) = 1 + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \mathrm{T}[H_I(t_1) H_I(t_2) \cdots H_I(t_n)]. \tag{8.29}$$

One can also write the expansion more compactly as

$$U_I(t, t_0) = \mathrm{T}\left[\exp\left(-\frac{i}{\hbar} \int_{t_0}^{t} dt' H_I(t')\right)\right], \tag{8.30}$$

where the operation of time ordering on an exponential is defined by its operation on the individual terms in the corresponding Taylor series.

In scattering processes, we are interested in how the states in the distant past evolve into other states in the distant future. We assume the interactions are localized in spacetime, that is, $H_I \to 0$ as $t \to \pm\infty$. Hence, the states $|\psi(t)\rangle_I \to |\psi(\pm\infty)\rangle_I$ at past and future infinities are the eigenstates of the non-interacting Hamiltonian H_0. We define the **S-matrix (scattering matrix)** as

$$S = \lim_{\substack{t \to +\infty, \\ t_0 \to -\infty}} U(t_0, t) \tag{8.31}$$

so that

$$|\psi(+\infty)\rangle_I = S\, |\psi(-\infty)\rangle_I\,. \tag{8.32}$$

We call $|\psi(-\infty)\rangle_I =: |i\rangle$ the initial (or incoming) states and $|\psi(+\infty)\rangle_I =: |f\rangle$ the final (or outgoing) states. The probability amplitude for this transition is the matrix element

$$A_{fi} = \langle f|S|i\rangle =: S_{fi}. \tag{8.33}$$

The conservation of probability demands that the S-matrix must be unitary, i.e. $S^\dagger S = 1$ because

$$\begin{aligned} 1 = \langle f|f\rangle = \langle i|S^\dagger S|i\rangle &= \sum_f \langle i|S^\dagger|f\rangle \langle f|S|i\rangle \\ \implies \quad \sum_f S^*_{if} S_{fi} &= 1. \end{aligned} \tag{8.34}$$

8.3 Wick's theorem

As we have seen, the S-matrix is expressed as a Taylor expansion of time-ordered products of the interaction Hamiltonian in the interaction picture:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} d^4x_1 \cdots d^4x_n \mathrm{T}[\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n)]. \tag{8.35}$$

How to calculate such expressions? It turns out that the *Wick's theorem* will help significantly simplify the calculations. To properly state the Wick's theorem, let us first define the **chronological contractions**. The chronological contraction of two bosonic operators is defined as

$$\overbrace{A(x)B(x')} = \begin{cases} \left[A^{(+)}(x), B^{(-)}(x')\right], & \text{if } x^0 > x'^0; \\ \left[B^{(+)}(x'), A^{(-)}(x)\right], & \text{if } x^0 < x'^0. \end{cases} \tag{8.36}$$

The chronological contraction between two fermionic operators is defined as

$$\overbrace{A(x)B(x')} = \begin{cases} \left\{A^{(+)}(x), B^{(-)}(x')\right\}, & \text{if } x^0 > x'^0; \\ -\left\{B^{(+)}(x'), A^{(-)}(x)\right\}, & \text{if } x^0 < x'^0. \end{cases} \tag{8.37}$$

Note that $(+)$ and $(-)$ denote the annihilation and creation operators, respectively. One can show that these two definitions really come from the Wick's theorem: $\overbrace{A(x)B(x')} = \mathrm{T}[A(x)B(x')] - \mathrm{N}[A(x)B(x')]$. Moreover, the normal-ordered product of field operators with contractions is defined as

$$\mathrm{N}\left[\overbrace{ABC...M}...DEF\right] = (-1)^q \overbrace{AM}\,\mathrm{N}[BC...DEF], \tag{8.38}$$

where q is the number of transpositions of fermionic operators which must be performed to go from $ABC...M...DEF$ to $AMBC...DEF$. Now we are ready to formulate the Wick's theorem.

Wick's Theorem. *Time-ordered product of field operators $A, B, C, \ldots$ is equal to the sum of normal-ordered operators with all possible chronological contractions among them, i.e.*

$$\begin{aligned} \mathrm{T}[ABCDEF...] =& \mathrm{N}[ABCDEF...]+ \\ & \mathrm{N}\left[\overbrace{AB}\,DEF...\right] + \mathrm{N}\left[\overbrace{ABC}\,DEF...\right] + \cdots + \\ & \mathrm{N}\left[\overbrace{AB}\,\overbrace{CD}\,EF...\right] + \mathrm{N}\left[\overbrace{AB}\,\overbrace{CDE}\,F...\right] + \cdots \end{aligned} \tag{8.39}$$

As we can see, in the first line we have just the normal ordering with contractions. In the second line we have all possible normal orderings with one contraction. And in the third line we have all possible normal orderings with two contractions, etc.

We will consider an example with two operators to illustrate the above theorem. Each field operator is the sum of two operators: the positive frequency operator which contains the annihilation operator, and the negative frequency operator which contains the creation operator. Consider the product of the operators $A(x)$ and $B(x')$:

$$\begin{aligned} A(x)B(x') &= \left[A^{(+)}(x) + A^{(-)}(x)\right] \cdot \left[B^{(+)}(x') + B^{(-)}(x')\right] \\ &= \mathrm{N}\left[A^{(+)}(x)B^{(+)}(x')\right] + \mathrm{N}\left[A^{(-)}(x)B^{(+)}(x')\right] + \mathrm{N}\left[A^{(-)}(x)B^{(-)}(x')\right] + A^{(+)}(x)B^{(-)}(x'). \end{aligned} \tag{8.40}$$

Remember that normal ordering puts all the creation operators to the left of annihilation operators. Therefore, the first three terms are automatically normal-ordered. Consider the last term, which can be written as

$$\begin{aligned} A^{(+)}(x)B^{(-)}(x') =& \mathrm{N}\left[A^{(+)}(x)B^{(-)}(x')\right] \\ &+ \begin{cases} \left[A^{(+)}(x), B^{(-)}(x')\right], & \text{if } A \text{ and } B \text{ are bosonic operators;} \\ \left\{A^{(+)}(x), B^{(-)}(x')\right\}, & \text{if } A \text{ and } B \text{ are fermionic operators.} \end{cases} \end{aligned} \tag{8.41}$$

Putting Eq. (8.41) into (8.40), we obtain

$$\begin{aligned} A(x)B(x') =& \mathrm{N}[A(x)B(x')] \\ &+ \begin{cases} \left[A^{(+)}(x), B^{(-)}(x')\right], & \text{if } A \text{ and } B \text{ are bosonic operators;} \\ \left\{A^{(+)}(x), B^{(-)}(x')\right\}, & \text{if } A \text{ and } B \text{ are fermionic operators.} \end{cases} \end{aligned} \tag{8.42}$$

Then the time-ordered product of these two field operators (assuming $x^0 > x'^0$) is just

$$\mathrm{T}[A(x)B(x')] = \mathrm{N}[A(x)B(x')] + \overbrace{A(x)B(x')}\,. \tag{8.43}$$

This is indeed the Wick's theorem for the product of two operators. Again, remember that time ordering (chronological ordering) places the operators defined at later times to the left of those defined at earlier times.

Coming back to the S-matrix, we know that it contains the sum of integrals with time-ordered products of interaction Hamiltonians. And we also know that in QFT, the interaction Hamiltonian are normal-ordered products of field operators. Thus in the S-matrix we typically have the products

$$\mathrm{T}[\mathrm{N}[ABC]...\mathrm{N}[DEF]], \tag{8.44}$$

where the field operators of any T-product have the same time argument. Such T-products are called the *mixed products*. In order to apply the Wick's theorem, we add an arbitrary small positive quantity ϵ to the time argument of all creation operators in the mixed T-product so that the normal ordering of the operators can be omitted and the mixed T-product becomes the ordinary T-product. At the end, we can set $\epsilon = 0$.
Note also that the contractions of operators which enter the mixed product under the N operator are always zero after adding positive ϵ to the time argument of all the creation operators, e.g. $\mathrm{T}[\mathrm{N}[\overbrace{A, B}]] \to \mathrm{T}[\overbrace{AB}] \Longrightarrow \overbrace{AB} = 0$. To see this, we write

$$\begin{aligned}\overbrace{A(x)B(x)} &= \overbrace{\Big[A^{(+)}(x^0, \mathbf{x}) + A^{(-)}(x^0 + \epsilon, \mathbf{x})\Big]\Big[B^{(+)}(x^0, \mathbf{x}) + B^{(-)}(x^0 + \epsilon, \mathbf{x})\Big]} \\ &= \overbrace{A^{(-)}(x^0 + \epsilon, \mathbf{x})B^{(+)}(x^0, \mathbf{x})} + \overbrace{A^{(+)}(x^0, \mathbf{x})B^{(-)}(x^0 + \epsilon, \mathbf{x})},\end{aligned} \tag{8.45}$$

where contractions of $A^{(+)}B^{(-)}$ and $A^{(-)}B^{(-)}$ always give zero from the Wick's theorem. Let us use the Wick's theorem to calculate the first term:

$$\begin{aligned}\overbrace{A^{(-)}(x^0 + \epsilon, \mathbf{x})B^{(+)}(x^0, \mathbf{x})} &= \mathrm{T}\Big[A^{(-)}(x^0 + \epsilon, \mathbf{x})B^{(+)}(x^0, \mathbf{x})\Big] - \mathrm{N}\Big[A^{(-)}(x^0 + \epsilon, \mathbf{x})B^{(+)}(x^0, \mathbf{x})\Big] \\ &= A^{(-)}(x^0 + \epsilon, \mathbf{x})B^{(+)}(x^0, \mathbf{x}) - A^{(-)}(x^0 + \epsilon, \mathbf{x})B^{(+)}(x^0, \mathbf{x}) \\ &= 0.\end{aligned} \tag{8.46}$$

Similarly, we have

$$\overbrace{A^{(+)}(x^0, \mathbf{x})B^{(-)}(x^0 + \epsilon, \mathbf{x})} = 0. \tag{8.47}$$

Therefore,

$$\overbrace{A(x)B(x)} = 0. \tag{8.48}$$

Hence, Wick's theorem can be applied to the mixed products (to compute the S-matrix), provided that one omits chronological contractions of the operators which are under the same normal ordering N and have the same space-time argument. As we see above, this is effectively equivalent to adding the small positive ϵ to the time argument of all creation operators.

9 Feynman Diagrams and Rules

Now we are ready to calculate the S-matrix element

$$A_{fi} = \langle f|S|i\rangle \tag{9.1}$$

in a perturbation theory (up to a given order n of the coupling constant) for a definite transition from an initial state $|i\rangle$ to a final state $|f\rangle$ and an interaction Hamiltonian $\mathcal{H}_I$. In general the S-matrix expansion results in many complicated transitions. However, only certain terms of the S-matrix contribute to a give transition $|i\rangle \to |f\rangle$. These terms must contain just the right annihilation operators to destroy particles in $|i\rangle$ and just the right creation operators to create the particles in $|f\rangle$. They usually also contain additional creation and annihilation operators which create and subsequently annihilate some particles. These particles are only present in the intermediate states and are called **virtual particles.**

According to the Wick's theorem, the time-ordered product of operators, such as those present in the expansion of the S-matrix, can be written as a sum of normal-ordered products of the operators where all creation operators stand to the left of annihilation operators. Such products of operators first annihilate a certain number of particles and then create some other particles. They do not cause creation and subsequent annihilation of virtual particles. Furthermore, each of these normal products effects a particular transition $|i\rangle \to |f\rangle$ which can be represented by a *Feynman diagram*. Actually there exists a one-to-one correspondence between the diagrams and the terms in the expansion (8.35), which can be summarized in simple *Feynman rules*.

Our goal is to calculate a few terms in the S-matrix expansion and try to determine the Feynman rules. We will take quantum electrodynamics (QED) as an example.

9.1 QED at $S^{(1)}$

Let us first write down the S-matrix expansion again:

$$S = \sum_{n=0}^{\infty} S^{(n)} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} d^4x_1 \cdots d^4x_n \mathrm{T}[\mathcal{H}_I(x_1)\cdots\mathcal{H}_I(x_n)]. \tag{9.2}$$

The interaction Hamiltonian density for QED is given by (refer to Eq. (8.13))

$$\mathcal{H}_I = e\,\mathrm{N}\big[\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\big]. \tag{9.3}$$

We can express all three field operators in terms of positive and negative frequency modes:

$$\begin{aligned} \psi &= \psi_+ + \psi_- \\ \overline{\psi} &= \overline{\psi}_+ + \overline{\psi}_- \\ A^\mu &= A^\mu_+ + A^\mu_-, \end{aligned} \tag{9.4}$$

where

$$\begin{aligned} &\psi_+ : \text{annihilation operator for } e^- \\ &\psi_- : \text{creation operator for } e^+ \\ &\overline{\psi}_+ : \text{annihilation operator for } e^+ \\ &\overline{\psi}_- : \text{creation operator for } e^- \\ &A^\mu_+ : \text{photon annihilation operator} \\ &A^\mu_- : \text{photon creation operator.} \end{aligned} \tag{9.5}$$

Then the $S^{(1)}$ term in the expansion Eq. (9.2) is

$$
\begin{aligned}
S^{(1)} &= -ie \int d^4x \, \mathrm{T}\big[\mathrm{N}\big[\overline{\psi}(x)\gamma_\mu \psi(x) A^\mu(x)\big]\big] \\
&= -ie \int d^4x \, \mathrm{N}\big[\overline{\psi}(x)\gamma_\mu \psi(x) A^\mu(x)\big] \qquad \text{(the chronological contraction within the same N is omitted)} \\
&= -ie \int d^4x \, \Big\{ \mathrm{N}\big[\overline{\psi}_+(x)\gamma_\mu \psi_+(x) A^\mu_+(x)\big] + \mathrm{N}\big[\overline{\psi}_+(x)\gamma_\mu \psi_+(x) A^\mu_-(x)\big] \\
&\qquad + \mathrm{N}\big[\overline{\psi}_+(x)\gamma_\mu \psi_-(x) A^\mu_+(x)\big] + \mathrm{N}\big[\overline{\psi}_+(x)\gamma_\mu \psi_-(x) A^\mu_-(x)\big] \\
&\qquad + \mathrm{N}\big[\overline{\psi}_-(x)\gamma_\mu \psi_+(x) A^\mu_+(x)\big] + \mathrm{N}\big[\overline{\psi}_-(x)\gamma_\mu \psi_+(x) A^\mu_-(x)\big] \\
&\qquad + \mathrm{N}\big[\overline{\psi}_-(x)\gamma_\mu \psi_-(x) A^\mu_+(x)\big] + \mathrm{N}\big[\overline{\psi}_-(x)\gamma_\mu \psi_-(x) A^\mu_-(x)\big] \Big\}.
\end{aligned}
\tag{9.6}
$$

Each normal-ordered product of fields corresponds to one Feynman diagram (not in order):

Figure 1: e^- scattering.

Figure 2: e^+ scattering.

Figure 3: e^+e^- annihilation.

Figure 4: e^+e^- creation.

However, note that none of the above processes contribute to the scattering of real physical particles for which we must have $k_\mu k^\mu = 0$ for the massless photons and $p_\mu p^\mu = m^2$ for the electrons and positrons with mass m. For example, let us consider the the electron-positron annihilation the the creation of a photon (Figure 3b). We can go from the position space to the momentum space by Fourier transform. In the initial state, there is an electron with momentum p_μ and polarization s and a positron with momentum p'_μ and polarization s':

$$|i\rangle = |\mathbf{p}'_{e^+}, s'; \mathbf{p}_{e^-}, s\rangle = d^\dagger_{s'}(\mathbf{p}')c^\dagger_s(\mathbf{p})\,|0\rangle\,. \tag{9.7}$$

In the final state we have a photon with momentum k_μ and polarization λ:

$$|f\rangle = |\mathbf{k}, \lambda\rangle = a^\dagger_\lambda(\mathbf{k})\,|0\rangle\,. \tag{9.8}$$

Then the transition amplitude is

$$\begin{aligned}
\langle f|S^{(1)}|i\rangle &= -ie\,\langle f|\int d^4x\;\mathrm{N}\big[\overline{\psi}_+(x)\gamma_\mu\psi_+(x)A^\mu_-(x)\big]|i\rangle \\
&= -ie\,\langle 0|a_\lambda(\mathbf{k})\int d^4x\;(x)A^\mu_-(x)\overline{\psi}_+(x)\gamma_\mu\psi_+(x)d^\dagger_{s'}(\mathbf{p}')c^\dagger_s(\mathbf{p})|0\rangle \\
&= -ie\int d^4x\int\!\!\int\!\!\int \frac{d^3\widetilde{k}}{(2\pi)^{3/2}\sqrt{2\omega_{\widetilde{\mathbf{k}}}}}\frac{d^3\widetilde{p}'}{(2\pi)^{3/2}\sqrt{2\omega_{\widetilde{\mathbf{p}}'}}}\frac{d^3\widetilde{p}}{(2\pi)^{3/2}\sqrt{2\omega_{\widetilde{\mathbf{p}}}}} \\
&\quad e^{i(\omega_{\widetilde{\mathbf{k}}}x^0-\widetilde{\mathbf{k}}\cdot\mathbf{x})}e^{-i(\omega_{\widetilde{\mathbf{p}}'}x^0-\widetilde{\mathbf{p}}'\cdot\mathbf{x})}e^{-i(\omega_{\widetilde{\mathbf{p}}}x^0-\widetilde{\mathbf{p}}\cdot\mathbf{x})}\sum_{\widetilde{\lambda}}\sum_{\widetilde{s}'}\sum_{\widetilde{s}}\epsilon^\mu_{\widetilde{\lambda}}(\widetilde{\mathbf{k}})\,\overline{v}_{\widetilde{s}'}(-\widetilde{\mathbf{p}}')u_{\widetilde{s}}(\widetilde{\mathbf{p}}) \\
&\quad \langle 0|a_\lambda(\mathbf{k})a^\dagger_{\widetilde{\lambda}}(\widetilde{\mathbf{k}})d_{\widetilde{s}'}(\widetilde{\mathbf{p}}')c_{\widetilde{s}}(\widetilde{\mathbf{p}})d^\dagger_{s'}(\mathbf{p}')c^\dagger_s(\mathbf{p})|0\rangle \\
&\quad \Big(a_\lambda(\mathbf{k})a^\dagger_{\widetilde{\lambda}}(\widetilde{\mathbf{k}}) = a^\dagger_{\widetilde{\lambda}}(\widetilde{\mathbf{k}})a_\lambda(\mathbf{k}) + \delta_{\lambda\widetilde{\lambda}}\delta^3(\mathbf{k}-\widetilde{\mathbf{k}}), \\
&\quad\;\; d_{\widetilde{s}'}(\widetilde{\mathbf{p}}')d^\dagger_{s'}(\mathbf{p}') = -d^\dagger_{s'}(\mathbf{p}')d_{\widetilde{s}'}(\widetilde{\mathbf{p}}') + \delta_{s'\widetilde{s}'}\delta^3(\mathbf{p}'-\widetilde{\mathbf{p}}'), \\
&\quad\;\; c_{\widetilde{s}}(\widetilde{\mathbf{p}})c^\dagger_s(\mathbf{p}) = -c^\dagger_s(\mathbf{p})c_{\widetilde{s}}(\widetilde{\mathbf{p}}) + \delta_{s\widetilde{s}}\delta^3(\mathbf{p}-\widetilde{\mathbf{p}}).\Big) \\
&= -ie\int \frac{d^4x}{(2\pi)^{9/2}\sqrt{8\omega_{\mathbf{k}}\omega_{\mathbf{p}}\omega_{\mathbf{k}'}}}e^{-i(\omega_{\mathbf{p}}+\omega_{\mathbf{p}'}-\omega_{\mathbf{k}})x^0}e^{i(\mathbf{p}+\mathbf{p}'-\mathbf{k})\cdot\mathbf{x}}\epsilon^\mu_\lambda(\mathbf{k})\overline{v}_{s'}(-\mathbf{p}')\gamma^\mu u_s(\mathbf{p}) \\
&= (2\pi)^4\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{p}'}-\omega_{\mathbf{k}})\delta^3(\mathbf{p}+\mathbf{p}'-\mathbf{k})\mathcal{M},
\end{aligned} \tag{9.9}$$

where

$$\mathcal{M} = \frac{\overline{v}_{s'}(-\mathbf{p}')}{(2\pi)^{3/2}\sqrt{2\omega_{\widetilde{\mathbf{p}}'}}}(-ie\gamma_\mu)\frac{u_s(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega_{\widetilde{\mathbf{p}}}}}\frac{\epsilon^\mu_\lambda(\mathbf{k}=\mathbf{p}+\mathbf{p}')}{(2\pi)^{3/2}\sqrt{2\omega_{\widetilde{\mathbf{k}}}}} \tag{9.10}$$

is called the **Feynman amplitude**.
A few comments are here. The δ-functions in the last line of Eq. (9.9) appear as a result of d^4x integration and reflect the conservation of energy and 3-momentum in the given process, $e^- + e^+ \to \gamma$:

$$\begin{aligned}
\omega_{\mathbf{p}} + \omega_{\mathbf{p}'} &= \omega_{\mathbf{k}} \\
\mathbf{p} + \mathbf{p}' &= \mathbf{k}.
\end{aligned} \tag{9.11}$$

However, it can be checked (with simple algebra) that this particular energy-momentum conservation is incompatible with the condition for real particles: $p_\mu p^\mu = p'_\mu p'^\mu = m^2$ and $k_\mu k^\mu = 0$ in our case. Thus, this process is simply unphysical. The same argument applies to all other processes at $S^{(1)}$.
More generally, we have

$$\langle f|S^{(n)}|i\rangle = 0 \tag{9.12}$$

for any unphysical process, i.e. a transition between real physical states which violates the conservation laws of the theory.

9.2 QED at $S^{(2)}$

To obtain real processes, we must consider at least the second-order term $S^{(2)}$ in the S-matrix expansion:

$$S^{(2)} = -\frac{e^2}{2}\int d^4x_1 \int d^4x_2 \mathrm{T}\big[\mathrm{N}[\overline{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)]\cdot\mathrm{N}[\overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)]\big]. \tag{9.13}$$

For the mixed product in $S^{(2)}$, according to the Wick's theorem (again, omitting the chronological contractions within the same normal ordering N), we have

$$\begin{aligned}
&\mathrm{T}\big[\mathrm{N}[\overline{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)]\mathrm{N}[\overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)]\big]\\
=&\mathrm{N}\big[\overline{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)\big]+\\
&\mathrm{N}\Big[\overbrace{\overline{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\overline{\psi}(x_2)\gamma_\nu\psi(x_2)}\,A^\nu(x_2)\Big]+\\
&\mathrm{N}\Big[\overline{\psi}(x_1)\gamma_\mu\,\overbrace{\psi(x_1)A^\mu(x_1)\overline{\psi}(x_2)}\,\gamma_\nu\psi(x_2)A^\nu(x_2)\Big]+\\
&\mathrm{N}\Big[\overline{\psi}(x_1)\gamma_\mu\psi(x_1)\,\overbrace{A^\mu(x_1)\overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)}\Big]+\\
&\mathrm{N}\Big[\overbrace{\overline{\psi}(x_1)\gamma_\mu\psi(x_1)\,\underbrace{A^\mu(x_1)\overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)}}\Big]+\\
&\mathrm{N}\Big[\overline{\psi}(x_1)\gamma_\mu\overbrace{\psi(x_1)\,\underbrace{A^\mu(x_1)\overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)}}\Big]+\\
&\mathrm{N}\Big[\overbrace{\overline{\psi}(x_1)\gamma_\mu\,\underbrace{\psi(x_1)A^\mu(x_1)\overline{\psi}(x_2)}\,\gamma_\nu\psi(x_2)}\,A^\nu(x_2)\Big]+\\
&\mathrm{N}\Big[\overbrace{\overline{\psi}(x_1)\gamma_\mu\overbrace{\psi(x_1)\,\underbrace{A^\mu(x_1)\overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)}}}\Big].
\end{aligned} \tag{9.14}$$

This first term again, does not lead to any real transition. The second and the third terms are identically equal to each other, which can be seen by permuting the field operators:

$$\mathrm{N}\Big[\overbrace{\overline{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)\overline{\psi}(x_2)\gamma_\nu\psi(x_2)}\,A^\nu(x_2)\Big] = \mathrm{N}\Big[\overline{\psi}(x_2)\gamma_\mu\,\overbrace{\psi(x_2)A^\mu(x_2)\overline{\psi}(x_1)}\,\gamma_\nu\psi(x_1)A^\nu(x_1)\Big]. \tag{9.15}$$

Hence, we can combine them and write

$$S_{\mathrm{I}}^{(2)} = -e^2\int d^4x_1\int d^4x_2\,\mathrm{N}\Big[\overline{\psi}(x_1)\gamma_\mu\,\overbrace{\psi(x_1)A^\mu(x_1)\overline{\psi}(x_2)}\,\gamma_\nu\psi(x_2)A^\nu(x_2)\Big], \tag{9.16}$$

which contains one fermion contraction. This chronological contraction is a c-number (classical number) and it is given by the fermion **propagator** which we will define later. It actually corresponds to a virtual fermion. In addition we have two uncontracted fermion operators and two uncontracted photon operators, corresponding to external particles in the initial and final states.
One of the processes described by $S_I^{(2)}$ is known as the **Compton scattering**:

$$\gamma + e^- \to \gamma + e^-. \tag{9.17}$$

This process corresponds to selecting the positive frequency part $\psi_+(x_2)$ of $\psi(x_2)$ to annihilate the initial electron and the negative frequency part $\overline{\psi}_-(x_1)$ of $\overline{\psi}(x_1)$ to create the final electron. But for the photons, we have two choices: we can take either $A_+^\mu(x_1)$ or $A_+^\mu(x_2)$ to annihilate the initial photon and correspondingly, $A_-^\mu(x_2)$ or $A_-^\mu(x_1)$ to create the final photon. Thus the Compton scattering S-matrix can be written as

$$S^{(2)}\big(\gamma e^- \to \gamma e^-\big) = S_a + S_b, \tag{9.18}$$

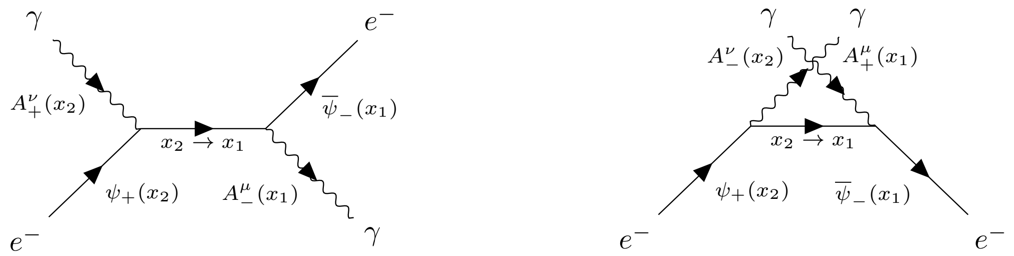
where

$$
\begin{aligned}
S_a &= -e^2 \int\int d^4x_1 d^4x_2\, \overline{\psi}_-(x_1)\gamma_\mu[iS_F(x_1-x_2)]\gamma_\nu A^\mu_-(x_1)A^\nu_+(x_2)\psi_+(x_2) \\
S_b &= -e^2 \int\int d^4x_1 d^4x_2\, \overline{\psi}_-(x_1)\gamma_\mu[iS_F(x_1-x_2)]\gamma_\nu A^\nu_-(x_2)A^\mu_+(x_1)\psi_+(x_2).
\end{aligned}
\tag{9.19}
$$

Note that

$$
iS_F(x_1-x_2) = \overbrace{\psi(x_1)\overline{\psi}(x_2)}
\tag{9.20}
$$

is the fermion propagator. The Feynman diagrams corresponding to S_a and S_b are shown below:



(a) S_a. (b) S_b.

Figure 5: The Feynman diagrams for Compton scattering.

Now we write the initial and final states as

$$
\begin{aligned}
|i\rangle &= |\mathbf{p}, s; \mathbf{k}, \lambda\rangle = c^\dagger_s(\mathbf{p})a^\dagger_\lambda(\mathbf{k})\,|0\rangle \\
|f\rangle &= |\mathbf{p}', s'; \mathbf{k}', \lambda'\rangle = c^\dagger_{s'}(\mathbf{p}')a^\dagger_{\lambda'}(\mathbf{k}')\,|0\rangle\,.
\end{aligned}
\tag{9.21}
$$

Hence the matrix element of S_a is

$$
\begin{aligned}
\langle f|S_a|i\rangle = -e^2 &\int d^4x_1 d^4x_2 \int \frac{d^3\widetilde{p}\; d^3\widetilde{k}\; d^3\widetilde{p}'\; d^3\widetilde{k}'}{(2\pi)^6\sqrt{16\omega_{\widetilde{\mathbf{p}}}\omega_{\widetilde{\mathbf{k}}}\omega_{\widetilde{\mathbf{p}}'}\omega_{\widetilde{\mathbf{k}}'}}} e^{i(\omega_{\widetilde{\mathbf{p}}'}x^0_1-\widetilde{\mathbf{p}}'\cdot\mathbf{x}_1)} e^{i(\omega_{\widetilde{\mathbf{k}}'}x^0_1-\widetilde{\mathbf{k}}'\cdot\mathbf{x}_1)} e^{-i(\omega_{\widetilde{\mathbf{k}}}x^0_2-\widetilde{\mathbf{k}}\cdot\mathbf{x}_2)} e^{-i(\omega_{\widetilde{\mathbf{p}}}x^0_2-\widetilde{\mathbf{p}}\cdot\mathbf{x}_2)} \\
&\sum_{\widetilde{s}'}\sum_{\widetilde{\lambda}'}\sum_{\widetilde{s}}\sum_{\widetilde{\lambda}} \overline{u}_{\widetilde{s}'}(\widetilde{\mathbf{p}}')\,\gamma_\mu \left[\frac{1}{(2\pi^4)}\int d^4q\; iS_F(q)e^{-i[q_0(x^0_1-x^0_2)-\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)]}\right]\gamma_\nu\; \epsilon^\mu_{\widetilde{\lambda}'}(\widetilde{\mathbf{k}}')\; \epsilon^\nu_{\widetilde{\lambda}}(\widetilde{\mathbf{k}})\; u_{\widetilde{s}}(\widetilde{\mathbf{p}}) \\
&\underbrace{\langle 0|a_{\lambda'}(\mathbf{k}')c_{s'}(\mathbf{p}')c^\dagger_{\widetilde{s}'}(\widetilde{\mathbf{p}}')a^\dagger_{\widetilde{\lambda}'}(\widetilde{\mathbf{k}}')a_{\widetilde{\lambda}}(\widetilde{\mathbf{k}})c_{\widetilde{s}}(\widetilde{\mathbf{p}})c^\dagger_s(\mathbf{p})a^\dagger_\lambda(\mathbf{k})|0\rangle}_{\propto\; \delta^3(\mathbf{k}'-\widetilde{\mathbf{k}}')\delta^3(\mathbf{p}-\widetilde{\mathbf{p}}')\delta^3(\mathbf{p}-\widetilde{\mathbf{p}})\delta^3(\mathbf{k}-\widetilde{\mathbf{k}})} \\
= -e^2 &\int \frac{d^4x_1\; d^4x_2}{(2\pi)^6\sqrt{16\omega_{\widetilde{\mathbf{p}}}\omega_{\widetilde{\mathbf{k}}}\omega_{\widetilde{\mathbf{p}}'}\omega_{\widetilde{\mathbf{k}}'}}} \int \frac{d^4q}{(2\pi)^4} e^{i[(\omega_{\mathbf{p}'}+\omega_{\mathbf{k}'}-q_0)x^0_1-(\mathbf{p}'+\mathbf{k}'-\mathbf{q})\cdot\mathbf{x}_1]} e^{-i[(\omega_{\mathbf{p}}+\omega_{\mathbf{k}}-q_0)x^0_2-(\mathbf{p}+\mathbf{k}-\mathbf{q})\cdot\mathbf{x}_2]} \\
&\overline{u}_{s'}(\mathbf{p}')\;\gamma_\mu\;[iS_F(q)]\;\epsilon^\mu_{\lambda'}(\mathbf{k}')\;\epsilon^\nu_\lambda(\mathbf{k})\;\gamma_\nu\;u_s(\mathbf{p}) \\
= &\frac{1}{(2\pi)^6\sqrt{16\omega_{\widetilde{\mathbf{p}}}\omega_{\widetilde{\mathbf{k}}}\omega_{\widetilde{\mathbf{p}}'}\omega_{\widetilde{\mathbf{k}}'}}} \int d^4q\;(2\pi)^4\delta(\omega_{\mathbf{p}'}+\omega_{\mathbf{k}'}-q_0)\;\delta^3(\mathbf{p}'+\mathbf{k}'-\mathbf{q})\;\delta(-\omega_{\mathbf{p}}-\omega_{\mathbf{k}}+q_0)\;\delta^3(-\mathbf{p}-\mathbf{k}+\mathbf{q}) \\
&\overline{u}_{s'}(\mathbf{p}')\;(-ie\gamma_\mu)\;\epsilon^\mu_{\lambda'}(\mathbf{k}')\;[iS_F(q)]\;\epsilon^\nu_\lambda(\mathbf{k})\;(-ie\gamma_\nu)\;u_s(\mathbf{p}) \\
= &(2\pi)^4\delta[(\omega_{\mathbf{p}'}+\omega_{\mathbf{k}'})-(\omega_{\mathbf{p}}+\omega_{\mathbf{k}})]\;\delta^3[(\mathbf{p}'+\mathbf{k}')-(\mathbf{p}+\mathbf{k})]\cdot\mathcal{M}_a,
\end{aligned}
\tag{9.22}
$$

where

$$
\mathcal{M}_a = \frac{\overline{u}_{s'}(\mathbf{p}')}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}'}}}(-ie\gamma_\mu)\frac{\epsilon^\mu_{\lambda'}(\mathbf{k}')}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}'}}}[iS_F(q_\mu = p_\mu + k_\mu)]\frac{\epsilon^\nu_\lambda(\mathbf{k})}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}}}}(-ie\gamma_\nu)\frac{u_s(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}}}}
\tag{9.23}
$$

is the Feynman amplitude for S_a process. Similarly, for S_b, the Feynman amplitude is

$$
\mathcal{M}_b = \frac{\overline{u}_{s'}(\mathbf{p}')}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}'}}}(-ie\gamma_\nu)\frac{\epsilon^\nu_\lambda(\mathbf{k})}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}'}}}\left[iS_F(q_\mu = p_\mu - k'_\mu)\right]\frac{\epsilon^\mu_{\lambda'}(\mathbf{k}')}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{k}'}}}(-ie\gamma_\mu)\frac{u_s(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}}}}.
\tag{9.24}
$$

Some other real processes described by $S_{\rm I}^{(2)}$ include the **Compton scattering by positrons**:

$$\gamma + e^+ \to \gamma + e^+, \tag{9.25}$$

the two-photon **pair annihilation**:

$$e^+ + e^- \to \gamma + \gamma, \tag{9.26}$$

and the **pair creation**:

$$\gamma + \gamma \to e^+ + e^-. \tag{9.27}$$

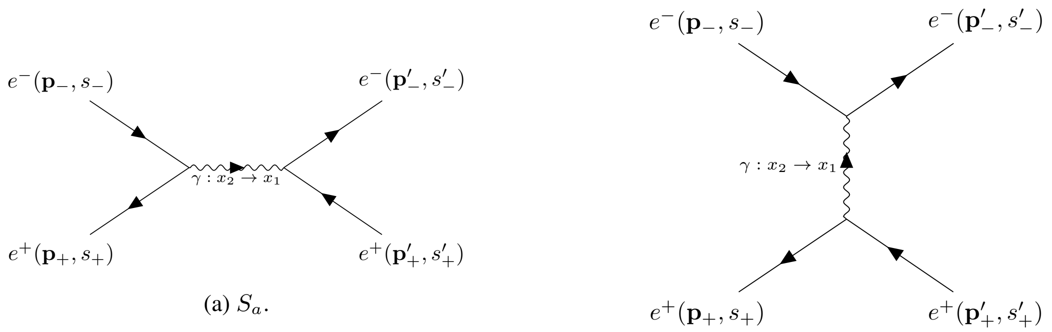
Now let us consider the fourth term in Eq.9.14, i.e. a single contraction of two photon fields. The S-matrix is

$$S_{\rm II}^{(2)} = -\frac{e^2}{2}\int d^4x_1 \int d^4x_2\, \mathrm{N}\Big[\overline{\psi}(x_1)\gamma_\mu\psi(x_1)\, \overbrace{A^\mu(x_1)\overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2)}\Big], \tag{9.28}$$

which describes the following processes: electron-electron scattering (**Møller scattering**), $e^- + e^- \to e^- + e^-$; positron-positron scattering, $e^+ + e^+ \to e^+ + e^+$; and electron-positron scattering (**Bhabha scattering**), $e^- + e^+ \to e^- + e^+$.
For Bhabha scattering, we have the following initial and final states:

$$\begin{aligned} |i\rangle &= |\mathbf{p}_-, s_-; \mathbf{p}_+, s_+\rangle = c^\dagger_{s_-}(\mathbf{p}_-) d^\dagger_{s_+}(\mathbf{p}_+)\,|0\rangle \\ |f\rangle &= |\mathbf{p}'_-, s'_-; \mathbf{p}'_+, s'_+\rangle = c^\dagger_{s'_-}(\mathbf{p}_-) d^\dagger_{s'_+}(\mathbf{p}'_+)\,|0\rangle\,. \end{aligned} \tag{9.29}$$

There are in total four contributions (four Feynman diagrams) to this process, but two of them are actually equivalent by simply exchanging the positions x_1 and x_2. The two (topologically) distinct Feynman diagrams are shown below.



(a) S_a.

(b) S_b.

Figure 6: The Feynman diagrams for Bhabha scattering:
(a). $(\mathbf{p}_-, s_-) - \psi_+(x_2),\ (\mathbf{p}_+, s_+) - \overline{\psi}_+(x_2),\ (\mathbf{p}'_-, s'_-) - \overline{\psi}_-(x_1),\ (\mathbf{p}'_+, s'_+) - \psi_-(x_1);$
(b). $(\mathbf{p}_-, s_-) - \psi_+(x_2),\ (\mathbf{p}_+, s_+) - \overline{\psi}_+(x_1),\ (\mathbf{p}'_-, s'_-) - \overline{\psi}_-(x_2),\ (\mathbf{p}'_+, s'_+) - \psi_-(x_1).$

Then again we have

$$S^{(2)}(e^-e^+ \to e^-e^+) = S_a + S_b, \tag{9.30}$$

where

$$\begin{aligned} S_a &= -e^2\int d^4x_1 d^4x_2\, \mathrm{N}\big[\overline{\psi}_-(x_1)\gamma_\mu\psi_-(x_1)\overline{\psi}_+(x_2)\gamma^\nu\psi_+(x_2)\big]\ [iD_F^{\mu\nu}(x_1-x_2)] \\ &= -e^2\int d^4x_1 d^4x_2\, \overline{\psi}_-(x_1)\gamma_\mu\psi_-(x_1)\overline{\psi}_+(x_2)\gamma^\nu\psi_+(x_2)[iD_F^{\mu\nu}(x_1-x_2)]; \\ S_b &= -e^2\int d^4x_1 d^4x_2\, \mathrm{N}\big[\overline{\psi}_+(x_1)\gamma_\mu\psi_-(x_1)\overline{\psi}_-(x_2)\gamma^\nu\psi_+(x_2)\big]\ [iD_F^{\mu\nu}(x_1-x_2)] \\ &= e^2\int d^4x_1 d^4x_2\, \overline{\psi}_-(x_2)\gamma_\mu\psi_-(x_1)\overline{\psi}_+(x_1)\gamma^\nu\psi_+(x_2)[iD_F^{\mu\nu}(x_1-x_2)]. \end{aligned} \tag{9.31}$$

Then we can calculate the matrix element $\langle f|S_a|i\rangle$ and $\langle f|S_b|i\rangle$ by following the same steps as Eq. (9.22). The two Feynman amplitudes are found to be

$$\mathcal{M}_a = \frac{\overline{u}_{s'_-}(\mathbf{p}'_-)}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}'_-}}}(-ie\gamma_\mu)\frac{v_{s'_+}(-\mathbf{p}'_+)}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}'_+}}}\left[iD_F^{\mu\nu}(q^\mu = p_-^\mu + p_+^\mu)\right]\frac{\overline{v}_{s_+}(-\mathbf{p}_+)}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}_+}}}(-ie\gamma_\nu)\frac{u_{s_-}(\mathbf{p}_-)}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}_-}}} \quad (9.32)$$

and

$$\mathcal{M}_b = -\mathcal{M}_a\,(p'_+ \leftrightarrow p_-,\ s'_+ \leftrightarrow s_-) = -\mathcal{M}_a\,(v_{s'_+}(-\mathbf{p}'_+) \leftrightarrow u_{s_-}(\mathbf{p}_-)). \quad (9.33)$$

Next is the second-order terms with more than one contraction. There are three terms with two contractions and one term with all field operators contracted. Note that the two terms in Eq. (9.14) with photon-photon and fermion-fermion contractions are equivalent when substituting into Eq. (9.13). So we can combine them and write

$$S^{(2)}_{\mathrm{III}} = -e^2 \int d^4x_1 \int d^4x_2\, \mathrm{N}\Big[\overline{\psi}(x_1)\gamma_\mu \overbrace{\psi(x_1)\, \underbrace{A^\mu(x_1)\overline{\psi}(x_2)}\gamma_\nu\psi(x_2)A^\nu(x_2)}\Big]. \quad (9.34)$$

This operator contains only two uncontracted fermionic operators, and therefore gives rise to two processes depending on whether it is electron or positron that is present in the initial and final states. For the electron case we have

$$S^{(2)}(e^- \to e^-) = -e^2 \int d^4x_1\, d^4x_2\, \overline{\psi}_-(x_1)\gamma_\mu[iS_F(x_1 - x_2)]\gamma_\nu\psi_+(x_2)[iD_F^{\mu\nu}(x_1 - x_2)]. \quad (9.35)$$

This expression corresponds to the Feynman diagram shown in Fig. 7. It represents one of the processes which

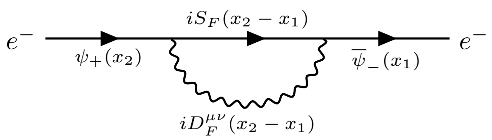


Figure 7: The electron self-energy diagram in the position space.

converts a bare electron into a physical electron, i.e. one surrounded by the photon cloud. This interaction changes the energy of the system, that is, the mass of the physical electron as compared with that of the bare electron. This is known as a **electron self-energy** term.
Let us calculate the Feynman amplitude for the electron self-energy diagram. In the initial and final states we have a single electron:

$$\begin{aligned} |i\rangle &= c_s^\dagger(\mathbf{p})\,|0\rangle \\ |f\rangle &= c_{s'}^\dagger(\mathbf{p}')\,|0\rangle\,. \end{aligned} \quad (9.36)$$

Then the transition amplitude is

$$\begin{aligned}
\langle f|S^{(2)}(e^- \to e^-)|i\rangle &= -e^2 \int d^4x_1\, d^4x_2 \int \frac{d^3\widetilde{p}'\, d^3\widetilde{p}}{(2\pi)^3\sqrt{4\omega_{\widetilde{\mathbf{p}}'}\omega_{\widetilde{\mathbf{p}}}}} \underbrace{e^{i(\omega_{\widetilde{\mathbf{p}}'}x_1^0 - \widetilde{\mathbf{p}}'\cdot\mathbf{x}_1)}}_{=e^{i\widetilde{p}'\cdot x_1}} \underbrace{e^{-i(\omega_{\widetilde{\mathbf{p}}}x_2^0 - \widetilde{\mathbf{p}}\cdot\mathbf{x}_2)}}_{=e^{-i\widetilde{p}\cdot x_2}} \sum_{\widetilde{s}'}\sum_{\widetilde{s}} \\
&\quad \overline{u}_{\widetilde{s}'}(\widetilde{\mathbf{p}}')\gamma_\mu \left[\int \frac{d^4q}{(2\pi)^4} iS_F(q) e^{-iq\cdot(x_1-x_2)}\right] \gamma_\nu u_{\widetilde{s}}(\widetilde{\mathbf{p}}) \left[\int \frac{d^4k}{(2\pi)^4} iD_F^{\mu\nu}(q) e^{-ik\cdot(x_1-x_2)}\right] \\
&\quad \langle 0|c_{s'}(\mathbf{p}')c^\dagger_{\widetilde{s}'}(\widetilde{\mathbf{p}}')c_{\widetilde{s}}(\widetilde{\mathbf{p}})c^\dagger_s(\mathbf{p}))|0\rangle \\
&= -e^2 \int \frac{d^4x_1\, d^4x_2}{(2\pi)^3\sqrt{\omega_{\mathbf{p}'}\omega_{\mathbf{p}}}} \int \frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} e^{i(p'-q-k)\cdot x_1} e^{i(-p+q+k)\cdot x_2} \\
&\quad \overline{u}_{s'}(\mathbf{p}')\gamma_\mu[iS_F(q)]\gamma_\nu u_s(\mathbf{p})[iD_F^{\mu\nu}(q)] \\
&= \frac{-e^2}{(2\pi)^3\sqrt{\omega_{\mathbf{p}'}\omega_{\mathbf{p}}}} \int d^4k\, d^4q\, \delta^4(p'-q-k)\delta^4(-p+q+k)\overline{u}_{s'}(\mathbf{p}')\gamma_\mu[iS_F(q)]\gamma_\nu u_s(\mathbf{p})[iD_F^{\mu\nu}(q)] \\
&= (2\pi)^4\delta^4(p'-p)\cdot\mathcal{M},
\end{aligned} \tag{9.37}$$

where

$$\mathcal{M} = \int \frac{d^4k}{(2\pi)^4}\, [iD_F^{\mu\nu}(k)]\, \overline{u}_{s'}(\mathbf{p}')\, (-ie\gamma_\mu)\, [iS_F(q=p-k)]\, (-ie\gamma_\nu)\, u_s(\mathbf{p}). \tag{9.38}$$

Therefore, the Feynman diagram in the momentum space is shown in Fig. 8.

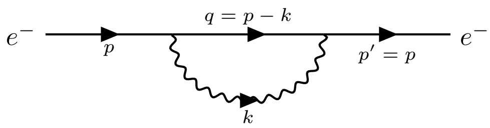


Figure 8: The electron self-energy diagram in the momentum space.

Notice that the Feynman amplitude has an integration over all internal photon momenta k, which is typical for a closed loop diagram. Moreover, this amplitude is divergent! These divergent self-energy effects can be eliminated by incorporating them into the properties of the physical electron. This process is known as the *renormalization*.

Finally, let us briefly discuss the term in $S^{(2)}$ with two fermion-fermion contractions. This term is called the **photon self-energy** term (because of an external photon present in the initial and final states, respectively) or the **vacuum polarization** term and has the following S-matrix:

$$S^{(2)}_{\text{IV}} = -e^2 \int d^4x_1 \int d^4x_2\, \mathrm{N}\left[\overbrace{\overline{\psi}(x_1)\gamma_\mu \underbrace{\psi(x_1)A^\mu_-(x_1)\overline{\psi}(x_2)}\gamma_\nu\psi(x_2)}\, A^\nu_+(x_2)\right], \tag{9.39}$$

which is described by the following Feynman diagram.

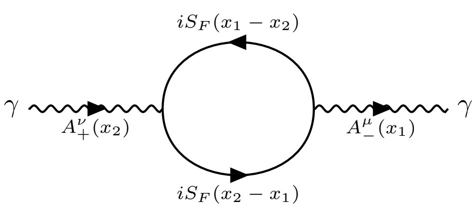


Figure 9: The photon self-energy diagram in the position space (left to right: $x_2 \to x_1$).

The normal product in $S_{\text{IV}}^{(2)}$ can be expressed as

$$\begin{aligned}
&\mathrm{N}\left[\overbrace{\overline{\psi}_\sigma(x_1)(\gamma_\mu)^{\sigma\rho}\,\underbrace{\psi_\rho(x_1)A_-^\mu(x_1)\overline{\psi}_\beta(x_2)}(\gamma_\nu)^{\beta\alpha}\psi_\alpha(x_2)}\,A_+^\nu(x_2)\right] \\
=&(-1)\,\overbrace{\psi_\alpha(x_2)\overline{\psi}_\sigma(x_1)}(\gamma_\mu)^{\sigma\rho}\,\overbrace{\psi_\rho(x_1)\overline{\psi}_\beta(x_2)}(\gamma_\nu)^{\beta\alpha}A_-^\mu(x_1)A_+^\nu(x_2) \\
=&(-1)\,\mathrm{Tr}\,[iS_F(x_2-x_1)\gamma_\mu iS_F(x_1-x_2)\gamma_\nu]A_-^\mu(x_1)A_+^\nu(x_2).
\end{aligned} \tag{9.40}$$

Notice that the minus sign and the trace in Eq. (9.40) are characteristics of a fermion loop. The trace basically corresponds to summing over all spin states of the virtual fermion-antifermion pair in the loop.

10 Propagators and Summary of Feynman Rules

To complete the derivation of Feynman rules in QED we have to calculate the chronological contractions for fermions

$$\overbrace{\psi(x_1)\overline{\psi}(x_2)} =: iS_F(x_1 - x_2) \tag{10.1}$$

and for photons

$$\overbrace{A^\mu(x_1)A^\nu(x_2)} =: iD_F^{\mu\nu}(x_1 - x_2). \tag{10.2}$$

10.1 Fermion propagators

Recall that the chronological contraction for fermions is given by Eq. (8.37)

$$\overbrace{\psi(x_1)\overline{\psi}(x_2)} = \begin{cases} \{\psi_+(x_1), \overline{\psi}_-(x_2)\}, & \text{if } x_1^0 > x_2^0; \\ -\{\overline{\psi}_+(x_2), \psi_-(x_1)\}, & \text{if } x_2^0 < x_1^0. \end{cases} \tag{10.3}$$

According to the Wick's theorem, the above chronological contraction can be written as

$$\overbrace{\psi(x_1)\overline{\psi}(x_2)} = \mathrm{T}\big[\psi(x_1)\overline{\psi}(x_2)\big] - \mathrm{N}\big[\psi(x_1)\overline{\psi}(x_2)\big]. \tag{10.4}$$

Hence if we compute the vacuum-to-vacuum transition amplitude for the operators for the operators in Eq. (10.4), we obtain

$$\begin{aligned} \overbrace{\psi(x_1)\overline{\psi}(x_2)} &=: iS_F(x_1 - x_2) \\ &= \langle 0|\mathrm{T}\big[\psi(x_1)\overline{\psi}(x_2)\big]|0\rangle - \underbrace{\langle 0|\mathrm{N}\big[\psi(x_1)\overline{\psi}(x_2)\big]|0\rangle}_{\to 0} \\ &= \langle 0|\mathrm{T}\big[\psi(x_1)\overline{\psi}(x_2)\big]|0\rangle\,, \end{aligned} \tag{10.5}$$

where in first step we have pulled the chronological contraction operator out: $\langle 0|\overbrace{\psi(x_1)\overline{\psi}(x_2)}|0\rangle = \overbrace{\psi(x_1)\overline{\psi}(x_2)}\,\langle 0|0\rangle = \overbrace{\psi(x_1)\overline{\psi}(x_2)}$. Thus the propagator $iS_F(x_1 - x_2)$ is nothing but the vacuum-to-vacuum transition amplitude of the time-ordered product of $\psi(x_1)$ and $\overline{\psi}(x_2)$ operators. We can expand the field operators in positive and negative frequency modes:

$$\begin{aligned} iS_F(x_1 - x_2) &= \langle 0|\mathrm{T}\big[\psi(x_1)\overline{\psi}(x_2)\big]|0\rangle \\ &= \langle 0|\mathrm{T}\big[\cancel{\psi_+(x_1)\overline{\psi}_+(x_2)} + \cancel{\psi_-(x_1)\overline{\psi}_-(x_2)} + \psi_+(x_1)\overline{\psi}_-(x_2) + \psi_-(x_1)\overline{\psi}_+(x_2)\big]|0\rangle \\ &= \begin{cases} \langle 0|\psi_+(x_1)\overline{\psi}_-(x_2)|0\rangle\,, & \text{if } x_1^0 > x_2^0; \\ -\langle 0|\overline{\psi}_+(x_2)\psi_-(x_1)|0\rangle\,, & \text{if } x_1^0 < x_2^0. \end{cases} \end{aligned} \tag{10.6}$$

The physical interpretation is clear: For $x_1^0 > x_2^0$, we can think of Eq. (10.6) as representing an electron created at x_2, traveling to x_1 and being annihilated there. Similarly, for $x_2^0 > x_1^0$, we can think of it as a positron created at x_1, propagating to x_2 and being annihilated there. Then the propagator can be rewritten as

$$iS_F(x_1 - x_2) = \theta(x_1^0 - x_2^0)iS_+(x_1 - x_2) - \theta(x_2^0 - x_1^0)iS_-(x_1 - x_2), \tag{10.7}$$

where

$$iS_+(x_1 - x_2) = \langle 0|\psi_+(x_1)\overline{\psi}_-(x_2)|0\rangle \tag{10.8}$$

and

$$iS_-(x_1 - x_2) = \langle 0|\overline{\psi}_+(x_2)\psi_-(x_1)|0\rangle\,. \tag{10.9}$$

$\theta(x)$ is again the unit step function defined in Eq. (8.27).
Now we can calculate S_+:

$$\begin{aligned}iS_+(x_1-x_2) &= \int \frac{d^3k\, d^3k'}{(2\pi)^3\sqrt{4\omega_{\mathbf{k}}\omega_{\mathbf{k'}}}}\sum_s\sum_{s'} u_s(\mathbf{k})\overline{u}_{s'}(\mathbf{k'})e^{-i(\omega_{\mathbf{k}}x_1^0-\mathbf{k}\cdot\mathbf{x}_1)}e^{i(\omega_{\mathbf{k'}}x_2^0-\mathbf{k'}\cdot\mathbf{x}_2)}\langle 0|c_s(\mathbf{k})c^\dagger_{s'}(\mathbf{k'})|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}}\sum_s u_s(\mathbf{k})\overline{u}_s(\mathbf{k})e^{-ik\cdot(x_1-x_2)} \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}}(\gamma^\mu k_\mu + m)e^{-ik_\mu(x_1-x_2)^\mu},\end{aligned} \tag{10.10}$$

where we have used $\sum_s u_s(\mathbf{k})\overline{u}_s(\mathbf{k}) = \gamma^\mu k_\mu + m$. Now using the fact that $k_\mu \to i\partial_\mu$ under the inverse Fourier transform (as in the above equation), we can write

$$iS_+(x) = (i\gamma^\mu\partial_\mu + m)[i\Delta_+(x)], \tag{10.11}$$

where

$$\Delta_+(x) = -i\int \frac{d^3k}{(2\pi)^3}\frac{e^{-i(\sqrt{\mathbf{k}^2+m^2}x^0-\mathbf{k}\cdot\mathbf{x})}}{2\sqrt{\mathbf{k}^2+m^2}}. \tag{10.12}$$

Again we have used the definition $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}$. Similarly, for $S_-(x)$, we use $\sum_s v_s(\mathbf{k})\overline{v}_s(\mathbf{k}) = -\gamma^\mu k_\mu + m$ to obtain

$$iS_-(x) = (i\gamma^\mu\partial_\mu + m)[i\Delta_-(x)], \tag{10.13}$$

where

$$\Delta_-(x) = -i\int \frac{d^3k}{(2\pi)^3}\frac{e^{i(\sqrt{\mathbf{k}^2+m^2}x^0-\mathbf{k}\cdot\mathbf{x})}}{2\sqrt{\mathbf{k}^2+m^2}}. \tag{10.14}$$

Hence, from Eq. (10.7) the propagator now becomes

$$iS_F(x) = \theta(x^0)(i\gamma^\mu\partial_\mu + m)[i\Delta_+(x)] - \theta(-x^0)(i\gamma^\mu\partial_\mu + m)[i\Delta_-(x)]. \tag{10.15}$$

To evaluate the propagator, we need to evaluate the integrals in Eqs. (10.12) and (10.14). By employing the contour integration method, we find that both integrals are equivalent to a four-dimensional integral given by

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int_{C_F} d^4k\,\frac{e^{-k_\mu x^\mu}}{k_\mu k^\mu - m^2}, \tag{10.16}$$

where C_F corresponds to the exact contour taken for integration over k_0 (different for $x^0 > 0$ and $x^0 < 0$). Note that this integral has two poles at $k = \pm\omega_{\mathbf{k}}$. To evaluate it, instead of deforming the contour, we can move the poles an infinitesimal distance away from the real axis, as shown in Fig. 10.

Figure 10: The displacements of poles for $\Delta_F(x)$.

Then the integral can be written as

$$\begin{aligned}\Delta_F(x) &= \frac{1}{(2\pi)^4}\int d^4k\,\frac{e^{-ik\cdot x}}{k_0^2 - (\omega_{\mathbf{k}} - i\eta)^2} \\ &= \frac{1}{(2\pi)^4}\int d^4k\,\frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon},\end{aligned} \tag{10.17}$$

where $\epsilon = 2\eta\omega_{\mathbf{k}}$ is a small positive number which we let tend to zero after integration.
It turns out that the function $i\Delta_F(x)$ we have obtained is actually the Feynman propagator for the **massive scalar**

field:

$$\overbrace{\phi(x_1)\phi(x_2)} = \langle 0|\mathrm{T}[\phi(x_1)\phi(x_2)]|0\rangle = i\Delta_F(x_1 - x_2). \tag{10.18}$$

For a complex scalar field, the propagator is

$$\overbrace{\phi(x_1)\phi^*(x_2)} = \langle 0|\mathrm{T}[\phi(x_1)\phi^*(x_2)]|0\rangle = i\Delta_F(x_1 - x_2). \tag{10.19}$$

In the momentum space, it can written as

$$i\Delta_F(k) = \frac{i}{k^2 - m^2 + i\epsilon}. \tag{10.20}$$

Now back to our **fermion propagator**, it can therefore be written as (using Eq. (10.15))

$$iS_F(x) = (i\gamma^\mu \partial_\mu + m)i\Delta_F(x) = i\int \frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\mu k_\mu + m}{k^2 - m^2 + i\epsilon} e^{-ik\cdot x}. \tag{10.21}$$

In the momentum space, it is

$$\begin{aligned} iS_F(k) &= i\,\frac{\gamma^\mu k_\mu + m}{k^2 - m^2 + i\epsilon} \\ &= \frac{i}{\gamma^\mu k_\mu - m + i\epsilon}, \end{aligned} \tag{10.22}$$

since $(\gamma^\mu k_\mu + m)(\gamma^\mu k_\mu - m) = k^2 - m^2$.

10.2 Photon propagators

In Lorentz (Feynman) gauge

In a similar way we can calculate the propagator for the photon field

$$\begin{aligned} \overbrace{A^\mu(x_1)A^\nu(x_2)} =: iD_F^{\mu\nu}(x_1 - x_2) &= \langle 0|\mathrm{T}[A^\mu(x_1)A^\nu(x_2)]|0\rangle \\ &= \theta(x_1^0 - x_2^0)iD_+^{\mu\nu}(x_1 - x_2) - \theta(x_2^0 - x_1^0)iD_-^{\mu\nu}(x_1 - x_2), \end{aligned} \tag{10.23}$$

where

$$iD_+^{\mu\nu}(x_1 - x_2) = \langle 0|A_+^\mu(x_1)A_-^\nu(x_2)|0\rangle \tag{10.24}$$

and

$$iD_-^{\mu\nu}(x_1 - x_2) = \langle 0|A_+^\nu(x_2)A_-^\mu(x_1)|0\rangle\,. \tag{10.25}$$

Then we calculate $D_+^{\mu\nu}$ as follows:

$$\begin{aligned} iD_+^{\mu\nu}(x_1 - x_2) &= \int \frac{d^3k\, d^3k'}{(2\pi)^3\sqrt{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} \sum_\lambda \sum_{\lambda'} \epsilon_\lambda^\mu(\mathbf{k})\epsilon_{\lambda'}^\nu(\mathbf{k}') e^{-i(\omega_{\mathbf{k}}x_1^0 - \mathbf{k}\cdot\mathbf{x}_1)} e^{i(\omega_{\mathbf{k}'}x_2^0 - \mathbf{k}'\cdot\mathbf{x}_2)} \langle 0|a_\lambda(\mathbf{k})a_{\lambda'}^\dagger(\mathbf{k}')|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_\lambda \epsilon_\lambda^\mu(\mathbf{k})\epsilon_\lambda^\nu(\mathbf{k}) e^{ik\cdot(x_1 - x_2)} \\ &= -\int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \eta^{\mu\nu} e^{ik_\mu(x_1 - x_2)^\mu}, \end{aligned} \tag{10.26}$$

where we have used

$$\sum_{\lambda=0}^{3} \epsilon_\lambda^\mu(\mathbf{k})\epsilon_\lambda^\nu(\mathbf{k}) = -\eta^{\mu\nu} \tag{10.27}$$

in the Lorenz gauge. Therefore, we can write

$$iD_+^{\mu\nu}(x) = -\eta^{\mu\nu} i\Delta_+(x), \tag{10.28}$$

where

$$\Delta_+(x) = \frac{1}{(2\pi)^4} \int_{C_+} d^4k \, \frac{e^{-ik_\mu x^\mu}}{k_\mu k^\mu}. \tag{10.29}$$

Similarly, for $D_-^{\mu\nu}$, we have

$$iD_-^{\mu\nu}(x) = -\eta^{\mu\nu} i\Delta_-(x), \tag{10.30}$$

where

$$\Delta_-(x) = \frac{1}{(2\pi)^4} \int_{C_-} d^4k \, \frac{e^{-ik_\mu x^\mu}}{k_\mu k^\mu}. \tag{10.31}$$

So now we can combine Eqs. (10.29) and (10.31) and write

$$\Delta_P(x) = \frac{1}{(2\pi)^4} \int_{C_P} d^4k \, \frac{e^{-ik_\mu x^\mu}}{k_\mu k^\mu} = \lim_{m\to 0} \Delta_F(x). \tag{10.32}$$

Again, by shifting the poles, we can write the integral above as

$$\Delta_P(x) = \frac{1}{(2\pi)^4} \int d^4k \, \frac{e^{-ik\cdot x}}{k^2 + i\epsilon}. \tag{10.33}$$

Thus, the **photon propagator in the Lorenz (Feynman) gauge** can be written as

$$iD_F^{\mu\nu}(x) = -\eta^{\mu\nu} i\Delta_P(x) = \frac{-i\eta^{\mu\nu}}{(2\pi)^4} \int d^4k \, \frac{e^{-ik\cdot x}}{k^2 + i\epsilon}. \tag{10.34}$$

In the momentum space, it is written as

$$iD_F^{\mu\nu}(k) = \frac{-i\eta^{\mu\nu}}{k^2 + i\epsilon}. \tag{10.35}$$

In Coulomb gauge

The derivation of the **photon propagator in the Coulomb gauge** is exactly the same as that in the Lorenz (Feynman) gauge. The only difference is that in the Coulomb gauge,

$$D_F^{00}(x_1 - x_2) = 0, \qquad D_F^{i0}(x_1 - x_2) = D_F^{0i}(x_1 - x_2) = 0. \tag{10.36}$$

And the sum of polarization vectors is (notice the difference from that in the Lorenz gauge)

$$\sum_{\lambda=1}^{2} \epsilon_\lambda^i(\mathbf{k}) \epsilon_\lambda^j(\mathbf{k}) = \delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}. \tag{10.37}$$

Therefore, the photon propagator is now written as

$$iD_F^{ij}(x) = \left(\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2}\right) i\Delta_P(x) = i\left(\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2}\right) \int \frac{d^4k}{(2\pi)^4} \, \frac{e^{-ik\cdot x}}{k^2 + i\epsilon}. \tag{10.38}$$

In the momentum space, it is

$$iD_F^{ij}(k) = \left(\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}\right) \frac{i}{k^2 + i\epsilon}. \tag{10.39}$$

10.3 Summary of Feynman rules for QED

As we have seen, for a transition $|i\rangle \to |f\rangle$, the S-matrix element (transition amplitude) is generally given by

$$\langle f|S|i\rangle = \delta_{fi} + (2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}_{fi}, \tag{10.40}$$

where the Feynman amplitude $\mathcal{M}_{fi}$ is given by

$$\mathcal{M}_{fi} = \sum_{n=1}^{\infty} \mathcal{M}_{fi}^{(n)}. \qquad (10.41)$$

The Feynman amplitude $\mathcal{M}_{fi}^{(n)}$ is obtained by drawing all topologically different Feynman diagrams in the momentum space which contain n vertices and the correct number of external lines. The contribution $\mathcal{M}_{fi}^{(n)}$ from each diagram is obtained from the following **Feynman rules** (for QED):

1. For each vertex, write a factor $(-ie\gamma_\mu)$.
2. For each internal photon line (photon propagator) labelled by momentum k, write a factor

$$iD_F^{\mu\nu} = \frac{-i\eta^{\mu\nu}}{k^2 + i\epsilon}.$$

3. For each internal fermion line (fermion propagator) labelled by momentum p, write a factor

$$iS_F(p) = i\frac{\gamma^\mu p_\mu + m}{p^2 - m^2 + i\epsilon} = \frac{i}{\gamma^\mu p_\mu - m + i\epsilon}.$$

4. For each external line, write one of the following factors:
   (a) For each initial electron, write
   $$\frac{u_s(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{p}}}.$$
   (b) For each final electron, write
   $$\frac{\overline{u}_s(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{p}}}.$$
   (c) For each initial positron, write
   $$\frac{\overline{v}_s(-\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{p}}}.$$
   (d) For each final positron, write
   $$\frac{v_s(-\mathbf{p})}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{p}}}.$$
   (e) For each initial/final photon, (both) write
   $$\frac{\epsilon_\lambda^\mu(\mathbf{k})}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{k}}}.$$
5. The spinor factors (γ-matrices, S_F-functions, four-spinors etc.) for each fermion line are ordered so that, reading from right to left, they occur in the same sequence as following the fermion line in the direction of its arrows.
6. For each closed fermion loop, take the trace and multiply it by a factor of (-1).
7. The 4-momenta associated with the three lines meeting at each vertex satisfy energy-momentum conservation. For each 4-momentum q which is not fixed by the the energy-momentum conservation, carry out the integral $\int \frac{d^4q}{(2\pi)^4}$. One such integration w.r.t. an internal 4-momentum q occurs for each closed loop.
8. Multiply the expression by a phase factor δ_p which is equal to $+1(-1)$ if an even (odd) number of interchanges of neighboring fermionic operators is required to perform the normal ordering.

10.4 Causality in QFT

There is a problem with *causality* in single-particle relativistic theory, i.e. in relativistic quantum mechanics. Consider the propagation of a free particle from a point x_0^i to a point x^i. This propagation is described by the transition amplitude $P(t) = \langle x^i | e^{-iHt} | x_0^i \rangle$. Now if we consider $|x^i| \gg t$ (i.e. well outside the light cone, acausal region), and do the integration (from inserting two complete sets of states $\int \frac{d^3k}{(2\pi)^3} |\mathbf{p}\rangle \langle \mathbf{p}|$ and $\int \frac{d^3k'}{(2\pi)^3} |\mathbf{p}'\rangle \langle \mathbf{p}'|$ into the transition amplitude), we will find that the probability for the acausal propagation is non-zero! Let us discuss this in details in QFT.

Consider, for simplicity, the case of real scalar fields. The amplitude for a corresponding particle to propagate from x_1 to x_2 is given by (NOT the Feynman propagator)

$$\begin{aligned} & \langle 0|\phi(x_2)\phi(x_1)|0\rangle \\ &= \langle 0|\phi_+(x_2)\phi_-(x_1)|0\rangle =: i\Delta_+(x_2 - x_1) \qquad (\phi_+ : a(\mathbf{k});\ \phi_- : a^\dagger(\mathbf{k})) \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i[\omega_\mathbf{k}\Delta t - \mathbf{k}\cdot(\mathbf{x}_2 - \mathbf{x}_1)]}}{2\omega_\mathbf{k}}, \end{aligned} \tag{10.42}$$

where $x_2^0 - x_1^0 = \delta t > 0$.
First consider where $x_2 - x_1$ is time-like (causal propagation), i.e. $(x_2 - x_1)_\mu (x_2 - x_1)^\mu > 0$. In this case we can always find the reference frame where $\mathbf{x}_2 - \mathbf{x}_1 = 0$ so that we can turn the three-dimensional integral into a one-dimensional integral:

$$\begin{aligned} i\Delta_+(x_2 - x_1) &= 4\pi \int_0^\infty \frac{|\mathbf{k}|^2 d|\mathbf{k}|}{(2\pi)^3} \frac{e^{-i\sqrt{|\mathbf{k}|^2 + m^2}\Delta t}}{2\sqrt{|\mathbf{k}|^2 + m^2}} \qquad (\int d^3k = 4\pi \int |\mathbf{k}|^2 d|\mathbf{k}| \text{ spherical integration}) \\ &= \frac{1}{4\pi^2} \int_0^\infty d\omega_\mathbf{k} \sqrt{\omega_\mathbf{k}^2 - m^2} e^{-i\omega_\mathbf{k}\Delta t} \qquad (d|\mathbf{k}| = \frac{\omega_\mathbf{k}\, d\omega_\mathbf{k}}{\sqrt{\omega_\mathbf{k}^2 - m^2}}) \\ &\sim e^{-im\Delta t} \qquad \text{as } \Delta t \to \infty, \end{aligned} \tag{10.43}$$

where in the last step, we have used the *stationary phase technique* to approximate the integral.
Now consider the acausal propagation where $\Delta t = 0$ and $\mathbf{x}_2 - \mathbf{x}_1 = \Delta\mathbf{x}$. Then the amplitude is

$$\begin{aligned} i\Delta_+(x_2 - x_1) &= \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\Delta\mathbf{x}}}{2\omega_\mathbf{k}} \\ &= \frac{1}{(2\pi)^3} \int |\mathbf{k}|^2 \sin\theta\, d\theta\, d\phi\, d|\mathbf{k}| \frac{e^{i|\mathbf{k}||\Delta\mathbf{x}|\cos\theta}}{2\sqrt{|\mathbf{k}|^2 + m^2}} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty d|\mathbf{k}| \frac{|\mathbf{k}|^2}{2\sqrt{|\mathbf{k}|^2 + m^2}} \frac{e^{i|\mathbf{k}||\Delta\mathbf{x}|} - e^{-i|\mathbf{k}||\Delta\mathbf{x}|}}{i|\mathbf{k}||\Delta\mathbf{x}|} \\ &= \frac{1}{(2i)(2\pi)|\Delta\mathbf{x}|} \int_{-\infty}^\infty d|\mathbf{k}| \frac{|\mathbf{k}| e^{i|\mathbf{k}||\Delta\mathbf{x}|}}{2\sqrt{|\mathbf{k}|^2 + m^2}} \\ &\sim e^{-m|\Delta\mathbf{x}|} \qquad \text{as } |\Delta\mathbf{x}| \to \infty. \end{aligned} \tag{10.44}$$

Again, we see that the acausal propagation of a particle is non-zero!
However, in QFT this acausal propagation cannot be measured in any real experiment. Indeed, the question we should ask is whether a measurement performed a tone point can affect a measurement at another point whose separation is space-like. Suppose one measures the field $\phi(x)$ at two different points x_2 and x_1. Then if the commutator $[\phi(x_2), \phi(x_1)]$ vanishes, this means that one measurement cannot affect the other. In fact, if the commutator $[\phi(x_2), \phi(x_1)] = 0$ for space-like separation, $(x_2 - x_1)^2 < 0$, then causality is preserved generally.
The proof that $[\phi(x_2), \phi(x_1)] = 0$ for $(x_2 - x_1)^2 < 0$ simply follows from the fact that this commutator is Lorentz invariant and that $[\phi(x_2), \phi(x_1)] = 0$ when $x_2^0 = x_1^0$ according to the basic equal-time commutation relations. We

can show that the commutator is indeed Lorentz invariant:

$$
\begin{aligned}
[\phi(x_2), \phi(x_1)] &= \int \frac{d^3k\, d^3k'}{(2\pi)^3\sqrt{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} \left[\left(a(\mathbf{k})e^{-i(\omega_{\mathbf{k}}x_2^0 - \mathbf{k}\cdot\mathbf{x}_2)} + a^\dagger(\mathbf{k})e^{i(\omega_{\mathbf{k}}x_2^0 - \mathbf{k}\cdot\mathbf{x}_2)} \right) + \left(a(\mathbf{k}')e^{-i(\omega_{\mathbf{k}'}x_1^0 - \mathbf{k}'\cdot\mathbf{x}_1)} + a^\dagger(\mathbf{k}')e^{i(\omega_{\mathbf{k}'}x_1^0 - \mathbf{k}'\cdot\mathbf{x}_1)} \right) \right] \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \left[e^{-ik\cdot(x_2 - x_1)} - e^{ik\cdot(x_2-x_1)} \right] \\
&= i\Delta_+(x_2 - x_1) - i\Delta_-(x_2 - x_1).
\end{aligned}
\tag{10.45}
$$

The $\Delta(x)$ in the equation above can be represented by a contour integral

$$
\Delta(x) = -\frac{i}{(2\pi)^4} \int_C d^4k\, \frac{e^{-ik\cdot x}}{k^2 - m^2}. \tag{10.46}
$$

Then $\Delta(x)$ is obviously Lorentz invariant.
We know that $[\phi(x_2), \phi(x_1)]_{x_2^0 = x_1^0} = 0$, and since this commutator is Lorentz invariant, it will vanish for any space-like interval. This is because we can always perform proper Lorentz transformations from the frame where $x_2^0 = x_1^0$ and thus

$$
[\phi(x_2), \phi(x_1)]_{x_2^0 = x_1^0} = 0 \qquad \Longrightarrow \qquad [\phi(x_2), \phi(x_1)] = 0 \quad \text{for any } (x_2 - x_1)^2 < 0. \tag{10.47}
$$

For the case of causal separation, $(x_2 - x_1)^2 \leqslant 0$, such a proper Lorentz transformation is impossible and $[\phi(x_2), \phi(x_1)] \neq 0$. In this way causality is preserved in QFT again.

11 Cross Section