Quantum Field Theory in Curved Spacetime

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While general relativity (GR) is a classical theory, we know that the world is fundamentally quantum mechanical. As the first step to a quantum theory of gravity, we may take the idea that quantized matter fields propagate on a fixed curved spacetime background, and study quantum field theory (QFT) in such a background. We shall see that even a naive attempt like this leads to some surprising results such as the Unruh effect and the Hawking radiation of black holes.

1 Quantum mechanics

In this section we give a quick reminder of canonical quantization of one of physicists' favorite systems, harmonic oscillators. A classical harmonic oscillator in 1D has the following Lagrangian:

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2, \tag{1}$$

where ω is a real constant and we have set the mass of the oscillator to unity for convenience. Using the Euler-Lagrange equation, we get the equation of motion

$$\ddot{q} + \omega^2 q = 0, \tag{2}$$

which admits a general solution

$$q(t) = a\mathrm{e}^{i\omega t} + a^*\mathrm{e}^{-i\omega t}, \tag{3}$$

where a is a complex constant. We may also define the canonical momentum

$$p \equiv \frac{\partial L(q, \dot{q})}{\partial \dot{q}} = \dot{q}, \tag{4}$$

and perform a Legendre transformation to find the Hamiltonian

$$H(p, q) = p\dot{q} - L = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2. \tag{5}$$

The Hamilton's equations of motion are

$$\dot{q} = p, \qquad \dot{p} = -\omega^2 q, \tag{6}$$

which are indeed equivalent to Eq. (2).

For a classical harmonic oscillator, we may identify the "ground state" as the state without motion, i.e. $q(t) = 0$ for all t, which is obviouslly the lowest-energy state of the system. On the other hand, the quantum theory of a harmonic oscillator is obtained by the standard procedure known as canonical quantization. Upon canonical quantization, we promote the classical coordinate $q(t)$ and momentum $p(t)$ to Hermitian operators (i.e. "hatted" objects) satisfying the same equations of motion (6), and demand that they satisfying the Heisenberg commutation relation[1]

$$[\hat{q}(t), \hat{p}(t)] = i\hbar. \tag{7}$$

In what follows, we will always set $\hbar = 1$. We note that the "classical ground state" $\hat{q}(t) = 0$ is impossible

[1] This postulate is required to realize the Heisenberg uncertainty principle.

for a quantum harmonic oscillator because the commutation relation cannot be satisfied by any $\hat{p}(t)$. Hence, a quantum oscillator cannot be completely at rest, and we shall find out what its lowest-energy state is. One way to obtain the energy levels of the harmonic oscillator is to go to the so-called Schrödinger picture, where the operators are time-independent and the state vectors or wavefunctions evolve in time. Since the Hamiltonian is time-independent in this picture, we can solve the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = \hat{H}\psi \tag{8}$$

by separation of variables, i.e. separating the wavefunctions into functions of the spatial coordinate and functions of time, $\psi(q,t) = f(t)g(q)$. The solutions are found to be (up to normalization)

$$\psi_n(q,t) = \mathrm{e}^{-\frac{1}{2}\omega q^2} H_n\big(\sqrt{\omega} q\big)\mathrm{e}^{-iE_n t}, \tag{9}$$

where H_n is the Hermite polynomial of degree n, and

$$E_n = \left(n + \frac{1}{2}\right)\omega. \tag{10}$$

The states $\psi_n(q,t)$ are eigenstates of the Hamiltonian, with energy eigenvalues E_n. Thus, we see that the energy of a quantum harmonic oscillator is quantized, and the ground state ($n = 0$) has energy $\omega/2$.

Another way of describing the quantum oscillators is to stay in the Heisenberg picture, where the operators are time-dependent and the quantum states are time-independent, and introduce the raising and lowering operators. This approach turns out to be more convenient for developing QFTs. We first define the raising operator $\hat{a}^+(t)$ and the lowering operator $\hat{a}^-(t)$ to be

$$\hat{a}^\pm(t) = \sqrt{\frac{\omega}{2}}\left[\hat{q}(t) \mp \frac{i}{\omega}\hat{p}(t)\right]. \tag{11}$$

It is easy to see that these two operators are Hermitian conjugate of each other, $[\hat{a}^-(t)]^\dagger = \hat{a}^+(t)$. The equations of motion of them can be derived from the Hamilton's equations (6):

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}^\pm(t) = \pm i\omega\hat{a}^\pm(t). \tag{12}$$

With the initial conditions, $\hat{a}^\pm(t)\big|_{t=0} = \hat{a}^\pm$, where $\hat{a}^\pm$ are time-independent operators which we will call creation and annihilation operators, respectively, the solutions are found to be

$$\hat{a}^\pm(t) = \hat{a}^\pm \mathrm{e}^{\pm i\omega t}. \tag{13}$$

We can readily rewrite the position and momentum operators in terms of the creation and annihilation operators as

$$\hat{q}(t) = \frac{1}{\sqrt{2\omega}}\big(\hat{a}^-\mathrm{e}^{-i\omega t} + \hat{a}^+\mathrm{e}^{i\omega t}\big), \qquad \hat{p}(t) = -i\sqrt{\frac{\omega}{2}}\big(\hat{a}^-\mathrm{e}^{-i\omega t} - \hat{a}^+\mathrm{e}^{i\omega t}\big). \tag{14}$$

Through the commutation relation (7), we find that the creation and annihilation operators satisfy the following commutation relation:

$$\left[\hat{a}^-, \hat{a}^+\right] = 1. \tag{15}$$

The Hamiltonian can be expressed as

$$\hat{H} = \frac{1}{2}\hat{p}(t)^2 + \frac{1}{2}\omega^2\hat{q}(t)^2 = \left(\hat{a}^+\hat{a}^- + \frac{1}{2}\right)\omega. \tag{16}$$

Assuming that the ground state exists and is unique, and the eigenvalues of $\hat{H}$ are bounded from below, the ground state $|0\rangle$ is found to satisfy

$$\hat{a}^-\left|0\right\rangle = 0, \tag{17}$$

i.e. the ground state is the one that can be annihilated away by the annihilation operator. Then we have $\hat{H}\left|0\right\rangle = \frac{1}{2}\omega\left|0\right\rangle$, that is, the ground state has the lowest energy $\omega/2$, which agrees with the Eq. (10). The excited states $|n\rangle$, where $n = 1, 2, \dots$, can be constructed by successive operation by the creation operators,

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^+\right)^n|0\rangle\,, \tag{18}$$

where the factors of $1/\sqrt{n!}$ are required by the orthonormality condition, $\langle m|n\rangle = \delta_{mn}$. It is straightforward to check that each state $|n\rangle$ is an eigenstate of the Hamiltonian,

$$\hat{H}\left|n\right\rangle = \underbrace{\left(n+\frac{1}{2}\right)\omega}_{E_n}\left|n\right\rangle. \tag{19}$$

2 QFT in flat spacetime

QFT is just a particular example of a quantum mechanical system, in which we quantize a field that permeate spacetime rather than a single oscillator. Here we consider the simplest example, a real scalar field $\phi(\mathbf{x},t)$ in 4d Minkowski spacetime. The action of such a scalar field is

$$S[\phi] = \int \mathrm{d}^4x\,\mathcal{L} = \int \mathrm{d}^4x\left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right], \tag{20}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric for a flat spacetime. By extremizing the action, the equation of motion is found to be

$$\partial_\mu\partial^\mu\phi - m^2\phi = \left(-\partial_t^2 + \nabla^2 - m^2\right)\phi = 0, \tag{21}$$

which is known as the Klein-Gordon equation. To solve this equation, we first perform a spatial Fourier decomposition on the scalar field,

$$\phi(\mathbf{x},t) = \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\phi_\mathbf{k}(t), \tag{22}$$

where the complex-valued functions $\phi_\mathbf{k}(t)$ are the Fourier modes that satisfy the relation $(\phi_\mathbf{k})^* = \phi_{-\mathbf{k}}$ because the field $\phi(\mathbf{x},t)$ is real. Substituting Eq. (22) into the Klein-Gordon equation, we find

$$\ddot{\phi}_\mathbf{k} + \underbrace{\left(|\mathbf{k}|^2 + m^2\right)}_{\equiv\omega_k^2}\phi_\mathbf{k} = 0. \tag{23}$$

In other words, each mode $\phi_\mathbf{k}(t)$ satisfies the equation of motion for a harmonic oscillator with frequency ω_k. And we know that the general solution is in the form of Eq. (3).

To quantize the scalar field, we similarly introduce the canonical momentum conjugate to the field ϕ,

$$\pi(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(\mathbf{x},t), \tag{24}$$

and promote both $\phi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ to operators that satisfy the equal-time commutation relation:

$$[\hat{\phi}(\mathbf{x},t),\hat{\pi}(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}); \quad [\hat{\phi}(\mathbf{x},t),\hat{\phi}(\mathbf{y},t)] = [\hat{\pi}(\mathbf{x},t),\hat{\pi}(\mathbf{y},t)] = 0. \tag{25}$$

Then the commutation relation for the mode functions (which are now operators) $\hat{\phi}_\mathbf{k}(t)$ and $\hat{\pi}_\mathbf{k}(t)$ can be readily obtained:

$$[\hat{\phi}_\mathbf{k}(t),\hat{\pi}_{\mathbf{k}'}(t)] = i\delta^3(\mathbf{k}+\mathbf{k}'). \tag{26}$$

Note that the plus sign in the delta function indicates that the variable that is conjugate to $\hat{\phi}_\mathbf{k}$ is not $\hat{\pi}_\mathbf{k}$

but $\hat{\pi}_{-\mathbf{k}} = \hat{\pi}_{\mathbf{k}}^{\dagger}$. For each mode $\hat{\phi}_{\mathbf{k}}(t)$, we proceed with quantization of a harmonic oscillator as in quantum mechanics. We introduce the time-dependent creation and annihilation operators:

$$\hat{a}_{\mathbf{k}}^{+}(t) = \sqrt{\frac{\omega_k}{2}}\left[\hat{\phi}_{-\mathbf{k}}(t) - \frac{i\hat{\pi}_{-\mathbf{k}}(t)}{\omega_k}\right], \qquad \hat{a}_{\mathbf{k}}^{-}(t) = \sqrt{\frac{\omega_k}{2}}\left[\hat{\phi}_{\mathbf{k}}(t) + \frac{i\hat{\pi}_{\mathbf{k}}(t)}{\omega_k}\right]. \tag{27}$$

The equations of motion for these operators are derived from Eq. (23):

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_{\mathbf{k}}^{\pm}(t) = \pm i\omega_k \hat{a}_{\mathbf{k}}^{\pm}(t), \tag{28}$$

whose solutions are

$$\hat{a}_{\mathbf{k}}^{\pm}(t) = \hat{a}_{\mathbf{k}}^{\pm}\mathrm{e}^{\pm i\omega_k t}, \tag{29}$$

where the time-independent creation and annihilation operators $\hat{a}_{\mathbf{k}}^{\pm}$ satisfy the commutation relations

$$[\hat{a}_{\mathbf{k}}^{-}, \hat{a}_{\mathbf{k}'}^{+}] = \delta^3(\mathbf{k} - \mathbf{k}'); \qquad [\hat{a}_{\mathbf{k}}^{-}, \hat{a}_{\mathbf{k}'}^{-}] = [\hat{a}_{\mathbf{k}}^{+}, \hat{a}_{\mathbf{k}'}^{+}] = 0. \tag{30}$$

The Hilbert space of the field states is built in a similar fashion as the quantum harmonic oscillator, although now we need to keep track of separate numbers of excitations for each momentum $\mathbf{k}$. We postulate the existence of the unique vacuum state $|0\rangle$ such that $\hat{a}_{\mathbf{k}}^{-}|0\rangle = 0, \forall\, \mathbf{k}$. Then a state with $n_{\mathbf{k}}$ particles (excitations) with the same momentum $\mathbf{k}$ is created by repeated action by the creation operator on the vacuum state,

$$|n_{\mathbf{k}}\rangle = \frac{1}{\sqrt{n_{\mathbf{k}}!}}\left(\hat{a}_{\mathbf{k}}^{+}\right)^{n_{\mathbf{k}}}|0\rangle\,, \tag{31}$$

while a state with n_i excitations of various momenta $\mathbf{k}_i$ is

$$|n_1, n_2, \cdots, n_j, \cdots\rangle = \frac{1}{\sqrt{n_1! n_2! \cdots n_j! \cdots}}\left(\hat{a}_{\mathbf{k}_1}^{+}\right)^{n_1}\left(\hat{a}_{\mathbf{k}_2}^{+}\right)^{n_2}\cdots\left(\hat{a}_{\mathbf{k}_j}^{+}\right)^{n_j}\cdots|0\rangle\,. \tag{32}$$

The Hilbert space of the field states is spanned by basis vectors $|n_1, n_2, \cdots\rangle$ with all possible choices of n_i. Such a space is called the Fock space.

Using Eqs. (27) and (29), the Fourier modes $\hat{\phi}_{\mathbf{k}}(t)$ can be expressed in terms of the time-independent creation and annihilation operators,

$$\hat{\phi}_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\omega_k}}\left(\hat{a}_{\mathbf{k}}^{-}\mathrm{e}^{-i\omega_k t} + \hat{a}_{\mathbf{k}}^{+}\mathrm{e}^{i\omega_k t}\right). \tag{33}$$

Thus, the complete mode expansion of the scalar field is

$$\begin{aligned}
\hat{\phi}(\mathbf{x}, t) &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left[\hat{a}_{\mathbf{k}}^{-}\mathrm{e}^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{-\mathbf{k}}^{+}\mathrm{e}^{i\omega_k t + i\mathbf{k}\cdot\mathbf{x}}\right] \\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left[\hat{a}_{\mathbf{k}}^{-}\mathrm{e}^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^{+}\mathrm{e}^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}}\right] \\
&= \int \mathrm{d}^3\mathbf{k}\left[\hat{a}_{\mathbf{k}}^{-} f_{\mathbf{k}}(\mathbf{x}, t) + \hat{a}_{\mathbf{k}}^{+} f_{\mathbf{k}}^{*}(\mathbf{x}, t)\right],
\end{aligned} \tag{34}$$

where the mode functions

$$f_{\mathbf{k}}(x^\mu) = \frac{\mathrm{e}^{ik_\mu x^\mu}}{\left[(2\pi)^3(2\omega)\right]^{1/2}} \tag{35}$$

and their complex conjugates $f_{\mathbf{k}}^{*}$ indeed form a complete orthonormal[2] set of solutions to the Klein-Gordon

[2] With respect to the inner product defined as an integral over a constant-time hypersurface Σ_t:

$$(\phi_1, \phi_2) = -i\int_{\Sigma_t}\mathrm{d}^3\mathbf{x}\,(\phi_1\partial_t\phi_2^{*} - \phi_2^{*}\partial_t\phi_1).$$

One can show that $(f_{\mathbf{k}_1}, f_{\mathbf{k}_2}) = \delta^3(\mathbf{k}_1 - \mathbf{k}_2)$ and $(f_{\mathbf{k}_1}^{*}, f_{\mathbf{k}_2}^{*}) = -\delta^3(\mathbf{k}_1 - \mathbf{k}_2)$.

equation. The $f_{\mathbf{k}}$ modes are called the positive-frequency modes, satisfying the Schrödinger equation

$$i\partial_t f_{\mathbf{k}} = \omega_k f_{\mathbf{k}}, \qquad \omega_k > 0, \tag{36}$$

while the $f_{\mathbf{k}}^*$ modes are called the negative-frequency modes satisfying

$$i\partial_t f_{\mathbf{k}}^* = -\omega_k f_{\mathbf{k}}^*, \qquad \omega_k > 0. \tag{37}$$

One crucial aspect of these modes in flat spacetime is our ability to distinguish between positive and negative frequencies, allowing for an interpretation of their an interpretation of their coefficients in the mode expansion of the scalar field as creation and annihilation operators. We consider another inertial observer in a Lorentz-boosted frame:

$$t' = \gamma t - \gamma \mathbf{v} \cdot \mathbf{x}, \qquad \mathbf{x}' = \gamma \mathbf{x} - \gamma \mathbf{v} t, \tag{38}$$

where $\gamma = 1/\sqrt{1 - |\mathbf{v}|^2}$. The inverse transformation is given by

$$t = \gamma t' + \gamma \mathbf{v} \cdot \mathbf{x}', \qquad \mathbf{x} = \gamma \mathbf{x}' + \gamma \mathbf{v} t'. \tag{39}$$

Then the time derivative of the mode functions in the boosted frame is

$$\begin{aligned} \partial_{t'} f_{\mathbf{k}} &= \frac{\partial x^\mu}{\partial t'} \partial_\mu f_{\mathbf{k}} \\ &= \gamma(-i\omega_k) f_{\mathbf{k}} + \gamma \mathbf{v} \cdot (i\mathbf{k}) f_{\mathbf{k}} \\ &= -i\omega_k' f_{\mathbf{k}}, \end{aligned} \tag{40}$$

or

$$i\partial_{t'} f_{\mathbf{k}} = \omega_k' f_{\mathbf{k}}, \tag{41}$$

where $\omega_k' = \gamma\omega_k - \gamma \mathbf{v} \cdot \mathbf{k}$ is simply the frequency in the boosted frame. Similarly, one finds $i\partial_{t'} f_{\mathbf{k}}^* = -\omega_k' f_{\mathbf{k}}^*$. Therefore, even though the frequency of a mode depends on the choice of inertial frame, the decomposition into positive and negative frequencies is invariant. Thus, any two inertial observers related by a Lorentz transformation in flat spacetime will agree on a unique set of creation and annihilation operators, which then uniquely determine the vacuum state.

3 QFT in curved spacetime

It is straightforward to generalize theories from flat to curved spacetime. We consider a minimally coupled (zero coupling to the curvature scalar R) real scalar field in a curved spacetime, whose action is now given by

$$S[\phi] = \int \mathrm{d}^4 x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 \right]. \tag{42}$$

The equation of motion is derived to be

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - m^2 \phi = 0, \tag{43}$$

which is a generalization of the Klein-Gordon equation in curved spacetime. To proceed, let us consider a specific example, a scalar field in the cosmological background, that is, we take the metric $g_{\mu\nu}$ to be the flat[3] FRW metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \mathrm{d}\mathbf{x}^2, \tag{44}$$

[3]It is only the three-dimensional spatial sections that are flat; the four-dimensional geometry of such spacetime is still curved.

where $a(t)$ is the scale factor. We may define the conformal time

$$\eta(t) \equiv \int_{t_0}^{t} \frac{\mathrm{d}t}{a(t)}, \tag{45}$$

where t_0 is an arbitrary constant. Then in the coordinates $(\eta, \mathbf{x})$, the FRW metric can be rewritten as

$$\mathrm{d}s^2 = a^2(\eta)\big(-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\big), \tag{46}$$

so the metric is conformally flat. Furthermore, it is convenient to introduce the auxiliary field $\chi \equiv a(\eta)\phi$. Then after some algebra the action (42) can be written as

$$S[\chi] = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3\mathbf{x}\Big[\big(\chi'\big)^2 - (\boldsymbol{\nabla}\chi)^2 - \tilde{m}^2(\eta)\chi^2\Big], \tag{47}$$

where $\chi' = \partial\chi/\partial\eta$, and we have denoted the time-dependent effective mass by $\tilde{m}$:

$$\tilde{m}^2(\eta) \equiv m^2a^2 - \frac{a''}{a}. \tag{48}$$

Thus, the dynamics of a scalar field ϕ in a flat FRW spacetime is mathematically equivalent to the dynamics of the auxiliary field χ in the Minkowski spacetime. All the information about the influence of gravity on the field ϕ is encapsulated in the time-dependent effective mass $\tilde{m}(\eta)$. It follows from the action that the equation of motion for the field $\chi(\eta, \mathbf{x})$ is

$$\chi'' - \nabla^2\chi + \tilde{m}^2(\eta)\chi = 0. \tag{49}$$

Expanding χ in Fourier modes,

$$\chi(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\chi_{\mathbf{k}}(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{50}$$

we find the equations of motion for the modes $\chi_{\mathbf{k}}(\eta)$ to be

$$\chi_{\mathbf{k}}'' + \underbrace{\Big[\tilde{m}^2(\eta) + |\mathbf{k}|^2\Big]}_{\equiv\omega_k^2(\eta)}\chi_{\mathbf{k}} = 0. \tag{51}$$

This corresponds to the equation of motion for a time-dependent harmonic oscillator. Recall from Quantum Mechanics I that the solution can be in general written in terms of different mode functions[4]. We begin by choosing the mode function $v_k(\eta)$, which is a complex-valued solution of

$$v_k'' + \omega_k^2(\eta)v_k = 0, \qquad \text{with} \quad \omega_k^2(\eta) \equiv \tilde{m}^2(\eta) + |\mathbf{k}|^2. \tag{52}$$

Then the general solution to Eq. (51) can be written as a linear combination of v_k and v_k^* as

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2}}\big[a_{\mathbf{k}}^- v_k^*(\eta) + a_{-\mathbf{k}}^+ v_k(\eta)\big], \tag{53}$$

[4]Eq. (51) has a two-dimensional space of solutions. Any two linearly independent solutions $x_1(\eta)$ and $x_2(\eta)$ form a basis in this space. It is easy to see that the Wronskian, $W[x_1, x_2] \equiv x_1'x_2 - x_1x_2'$, is time-independent if $x_{1,2}(\eta)$ are solutions of Eq. (51) and moreover, $W[x_1, x_2] \neq 0$ iff $x_1(\eta)$ and $x_2(\eta)$ are linearly independent solutions. If $\{x_1(\eta), x_2(\eta)\}$ is a basis of solutions, it is convenient to define the complex function $v(\eta) \equiv x_1(\eta) + ix_2(\eta)$ such that $v(\eta)$ and $v^*(\eta)$ are linearly independent and form a basis in the space of complex solutions. To quantize the time-dependent harmonic oscillator, if we choose to write

$$\hat{q}(\eta) = \frac{1}{\sqrt{2}}\big[\hat{a}^+v(\eta) + \hat{a}^-v^*(\eta)\big], \qquad \hat{p}(\eta) = \hat{q}'(\eta) = \frac{1}{\sqrt{2}}\big[\hat{a}^+v'(\eta) + \hat{a}^-v^{*\prime}(\eta)\big],$$

and impose the normalization condition on the mode functions: $\mathrm{Im}(v'v^*) = \frac{1}{2i}W[v, v^*] = 1$, then the canonical commutation relation $[\hat{q}(\eta), \hat{p}(\eta)] = i$ will yield the standard commutation relation for $\hat{a}^\pm$: $[\hat{a}^-, \hat{a}^+] = 1$.

where $a_{\mathbf{k}}^{\pm}$ are complex coefficients. Note that the index $-\mathbf{k}$ in the second coefficient and the factor $1/\sqrt{2}$ are chosen for later convenience. Then the mode expansion of the field $\chi(\eta,\mathbf{x})$ with respect to the mode functions $v_k(\eta)$ can be readily written as

$$\begin{aligned}\chi(\eta,\mathbf{x}) &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left[a_{\mathbf{k}}^{-}v_k^*(\eta)+a_{-\mathbf{k}}^{+}v_k(\eta)\right]\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left[a_{\mathbf{k}}^{-}v_k^*(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}+a_{\mathbf{k}}^{+}v_k(\eta)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\right].\end{aligned} \tag{54}$$

To quantize the field, we again promote the coefficients to time-independent operators $\hat{a}_{\mathbf{k}}^{\pm}$ and postulate the canonical commutation relation

$$[\hat{\chi}(\eta,\mathbf{x}),\hat{\chi}'(\eta,\mathbf{y})]=i\delta^3(\mathbf{x}-\mathbf{y}). \tag{55}$$

By imposing the normalization condition on the mode functions

$$\mathrm{Im}(v_k'v_k^*)=\frac{v_k'v_k^*-v_kv_k^{*\prime}}{2i}=\frac{W[v_k,v_k^*]}{2i}=1, \tag{56}$$

we obtain the usual commutation relations for the creation and annihilation operators,

$$[\hat{a}_{\mathbf{k}}^{-},\hat{a}_{\mathbf{k}'}^{+}]=\delta^3(\mathbf{k}-\mathbf{k}'); \qquad [\hat{a}_{\mathbf{k}}^{-},\hat{a}_{\mathbf{k}'}^{-}]=[\hat{a}_{\mathbf{k}}^{+},\hat{a}_{\mathbf{k}'}^{+}]=0. \tag{57}$$

Therefore, once the operators $\hat{a}_{\mathbf{k}}^{\pm}$ are determined, the vacuum state $|0\rangle$ is defined as the eigenstate of all annihilation operators with eigenvalue 0, i.e. $\hat{a}_{\mathbf{k}}^{-}|0\rangle=0,\forall\,\mathbf{k}$. All the excited states can be constructed in the same way as Eq. (32).

However, as mentioned above, the basis of the solutions to the time-dependent harmonic oscillators is not unique. We may choose another set of linearly independent solutions, $u_k(\eta)$ and $u_k^*(\eta)$, as our mode functions. The "old" mode function $v_k(\eta)$ can be in general written as a linear combination of the "new" mode functions $u_k(\eta)$ and $u_k^*(\eta)$ (or vice versa),

$$v_k(\eta)=\alpha_k u_k(\eta)+\beta_k u_k^*(\eta), \tag{58}$$

with complex coefficients α_k and β_k. If both sets $v_k(\eta)$ and $u_k(\eta)$ are normalized according to Eq. (56), then the coefficients satisfy

$$|\alpha_k|^2-|\beta_k|^2=1. \tag{59}$$

In canonical quantization, using the mode functions $u_k(\eta)$, one obtains an alternative mode expansion which defines another set of creation and annihilation operators, $\hat{b}_{\mathbf{k}}^{\pm}$:

$$\hat{\chi}(\eta,\mathbf{x})=\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left[\hat{b}_{\mathbf{k}}^{-}u_k^*(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}+\hat{b}_{\mathbf{k}}^{+}u_k(\eta)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{60}$$

Since both Eqs. (54) and (60) describe the mode expansion of the same field $\hat{\chi}(\eta,\mathbf{x})$, they must agree, i.e.

$$\begin{aligned}\hat{\chi}(\eta,\mathbf{x}) &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left[\hat{a}_{\mathbf{k}}^{-}v_k^*(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}+\hat{a}_{\mathbf{k}}^{+}v_k(\eta)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\right] \\ &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left\{\hat{a}_{\mathbf{k}}^{-}[\alpha_k^*u_k^*(\eta)+\beta_k^*u_k(\eta)]\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}+\hat{a}_{\mathbf{k}}^{+}[\alpha_ku_k(\eta)+\beta_ku_k^*(\eta)]\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\right\} \\ &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left[\left(\alpha_k^*\hat{a}_{\mathbf{k}}^{-}+\beta_k\hat{a}_{-\mathbf{k}}^{+}\right)u_k^*(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}+\left(\alpha_k\hat{a}_{\mathbf{k}}^{+}+\beta_k^*\hat{a}_{-\mathbf{k}}^{-}\right)u_k(\eta)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\right] \\ &\overset{!}{=} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left[\hat{b}_{\mathbf{k}}^{-}u_k^*(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}+\hat{b}_{\mathbf{k}}^{+}u_k(\eta)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\right].\end{aligned} \tag{61}$$

This immediately implies the following relations between the operators $\hat{b}^{\pm}_{\mathbf{k}}$ and $\hat{a}^{\pm}_{\mathbf{k}}$:

$$\hat{b}^{-}_{\mathbf{k}} = \alpha^*_k \hat{a}^{-}_{\mathbf{k}} + \beta_k \hat{a}^{+}_{-\mathbf{k}}, \qquad \hat{b}^{+}_{\mathbf{k}} = \alpha_k \hat{a}^{+}_{\mathbf{k}} + \beta^*_k \hat{a}^{-}_{-\mathbf{k}}. \tag{62}$$

Such relations are called the Bogolyubov transformation and the complex coefficients α_k, β_k are called the Bogolyubov coefficients. Furthermore, the two sets of annihilation operators $\hat{a}^{-}_{\mathbf{k}}$ and $\hat{b}^{-}_{\mathbf{k}}$ define their corresponding vacua, which we will call the a-vacuum and b-vacuum, respectively. Let us calculate the mean number of b-particles of the mode $\chi_{\mathbf{k}}$ in the a-vacuum state, which is given by the expectation value of the b-particle number operator $\hat{N}^{(b)}_{\mathbf{k}} \equiv \hat{b}^{+}_{\mathbf{k}} \hat{b}^{-}_{\mathbf{k}}$ in the state $|0_a\rangle$:

$$\begin{aligned}
\langle 0_a | \hat{b}^{+}_{\mathbf{k}} \hat{b}^{-}_{\mathbf{k}} | 0_a \rangle &= \langle 0_a | (\alpha_k \hat{a}^{+}_{\mathbf{k}} + \beta^*_k \hat{a}^{-}_{-\mathbf{k}})(\alpha^*_k \hat{a}^{-}_{\mathbf{k}} + \beta_k \hat{a}^{+}_{-\mathbf{k}}) | 0_a \rangle \\
&= \langle 0_a | \beta^*_k \beta_k \hat{a}^{-}_{-\mathbf{k}} \hat{a}^{+}_{-\mathbf{k}} | 0_a \rangle \\
&= |\beta_k|^2 \langle 0_a | \delta^3(0) + \hat{a}^{+}_{-\mathbf{k}} \hat{a}^{-}_{-\mathbf{k}} | 0_a \rangle \\
&= |\beta_k|^2 \delta^3(0).
\end{aligned} \tag{63}$$

The divergent factor $\delta^3(0)$ is a consequence of considering an infinite spatial volume. If we quantize the field in a box of finite volume, the divergent factor will be replaced by the box volume. Therefore, we obtain the mean density of b-particles in the a-vacuum:

$$n_k = |\beta_k|^2, \tag{64}$$

which is generally non-zero!

Recall that in flat spacetime, we were able to pick a natural set of positive- and negative-frequency mode functions by solving the time-independent harmonic oscillator (for a real scalar field). And under Lorentz transformations, all the positive-frequency modes stay positive-frequency, and all the negative-frequency modes stay negative-frequency. This implies that the creation and annihilation operators are the same in all inertial frames. Thus, every inertial observer will agree on what the vacuum state is, and how many particles are around. This boils down to the existence of a timelike Killing vector ∂_t in Minkowski spacetime and all such Killing vectors are related by Lorentz transformations. The mode functions are eigenfunctions of this Killing vector. But in the case of curved spacetime, there is generically no timelike Killing vector to define positive- and negative-frequency modes, and there is an ambiguity in the choice of mode basis. These different mode bases define different sets of creation and annihilation operators that are related by the Bogolyubov transformations, which in turn define different vacua. In particular, as we saw, if one inertial observer defines particles with respect to one set of modes v_k and another observer uses a different set of modes u_k, they will typically disagree on how many particles are observed.

4 Unruh effect and Hawking radiation

The Unruh effect (1976) is manifested even in flat spacetime. It predicts that a uniformly accelerating observer in the Minkowski vacuum state will observe a thermal spectrum of particles, while an inertial observer would observe none. By explicit calculation it can be shown that the density of massless scalar particles as seen by an accelerating observer follows the Bose-Einstein distribution, which is characteristic of a thermal blackbody radiation with a temperature called the Unruh temperature. On the other hand, the Hawking radiation (1974) is the blackbody radiation emitted by a static black hole (BH), as registered by a stationary observer far away from the BH horizon. The two effects are tightly related to each other. In fact, the Hawking radiation is just the Unruh effect with the equivalence principle applied to the BH horizons. Here we will present an argument that leads to the Unruh temperature in Minkowski geometry and the Hawking temperature in Schwarzschild geometry without doing any explicit calculation.

Recall that the BH geometry is described by the Schwarzschild metric

$$\mathrm{d}s^2 = -f \mathrm{d}t^2 + \frac{1}{f} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin\theta^2 \mathrm{d}\phi^2 \right), \tag{65}$$

where

$$f = 1 - \frac{2GM}{r} = 1 - \frac{r_s}{r}, \tag{66}$$

with r_s being the radius of the event horizon. Recall that there are two intrinsic geometric quantities associated with the BH horizon: one is the area of a spatial section,

$$A = 4\pi r_s^2 = 16\pi G^2 M^2, \tag{67}$$

and the other is the surface gravity,

$$\kappa = \frac{1}{2} f'(r_s) = \frac{1}{4GM}. \tag{68}$$

In a static, asymptotically flat spacetime, the surface gravity is the acceleration of a static observer near the horizon, as measured by a static observer at infinity.

Let us first consider the region near (but outside) the BH horizon. We may Taylor expand f around $r = r_s$:

$$f(r) \simeq \underbrace{f(r_s)}_{=0} + f'(r_s)(r - r_s) = f'(r_s)(r - r_s). \tag{69}$$

Then we introduce the proper distance ρ from the horizon:

$$\mathrm{d}\rho = \frac{\mathrm{d}r}{\sqrt{f}} \overset{r \to r_s}{=} \frac{\mathrm{d}r}{\sqrt{f'(r_s)(r - r_s)}}. \tag{70}$$

Integrating both sides gives

$$\rho = \frac{2}{\sqrt{f'(r_s)}} \sqrt{r - r_s}. \tag{71}$$

When $r = r_s$, $\rho = 0$, corresponding to the horizon. So we can express f in terms of the proper distance ρ as

$$f(r) = f'(r_s)(r - r_s) = \left[\frac{1}{2} f'(r_s)\right]^2 \rho^2 = \kappa^2 \rho^2. \tag{72}$$

Then the Schwarzschild metric near the horizon becomes

$$\begin{aligned} \mathrm{d}s^2 &= -\kappa^2 \rho^2 \mathrm{d}t^2 + \mathrm{d}\rho^2 + r_s^2 \mathrm{d}\Omega_2^2 \\ &= -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2 + r_s^2 \mathrm{d}\Omega_2^2, \end{aligned} \tag{73}$$

where we have defined $\eta = \kappa t = \frac{t}{2r_s}$. You may recognize that the first two terms in the above expression is the (1+1)d Minkowski metric in the Rindler form[5]. To see this, consider the 2d Minkowski spacetime

$$\mathrm{d}s^2_{\mathcal{M}_2} = -\mathrm{d}T^2 + \mathrm{d}X^2. \tag{74}$$

If we let $X = \rho \cosh \eta, T = \rho \sinh \eta$, we get

$$\mathrm{d}s^2_{\mathcal{M}_2} = -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2, \tag{75}$$

which is exactly the 2d Rindler spacetime. But since $X^2 - T^2 = \rho^2 \geq 0$, the Rindler coordinates only covers the $X \geq 0$ part of $\mathcal{M}_2$, i.e. region I as shown in Fig. 1. Note that the BH horizon $\rho = 0$ is mapped to the light cones $X = \pm T$, just like what we saw for the Schwarzschild geometry in the Kruskal coordinates (cf. right panel in Fig. 1). Then the near-horizon BH geometry can be viewed as *Rindler* $\times S^2$.

A few remarks are in order. First, an observer at $r =$ const. ($r \gtrsim r_s$) is mapped to an observer with $\rho =$ const. in a Rindler patch, that is, an observer in Minkowski spacetime following a hyperbolic trajectory, $X^2 - T^2 = \rho^2 =$ const. This corresponds to a uniformly accelerating observer in Minkowski spacetime.

[5] The Rindler coordinates serve as a proper frame for a uniformly accelerating observer, in which the proper time measured by the accelerated observer coincides with the coordinate time.

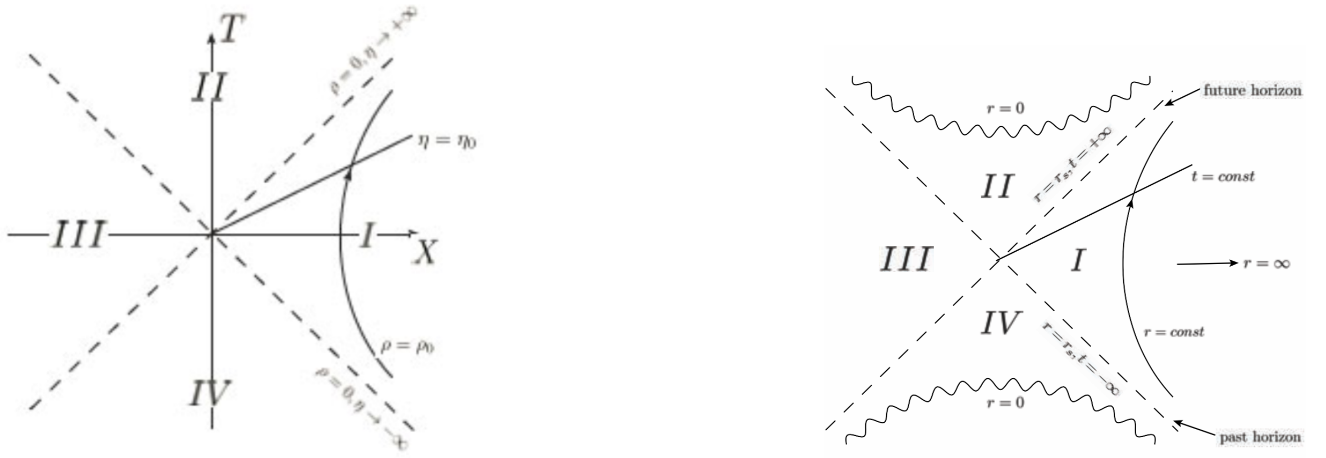

Figure 1: Left: Causal structure of 2d Minkowski spacetime in the Rindler form. Right: Causal structure of the full Schwarzschild geometry.

Indeed, one can check that such an observer has a constant proper acceleration[6] given by

$$a \equiv |\mathbf{a}| = \frac{1}{\rho} = \frac{\sqrt{f'(r_s)}}{2}\frac{1}{\sqrt{r-r_s}}. \tag{76}$$

And furthermore, the acceleration seen by an observer at infinity O_∞ would be

$$a_\infty = a(r)\sqrt{f(r)} = \frac{1}{2}f'(r_s) = \kappa, \tag{77}$$

which is exactly our interpretation of the surface gravity previously.

Now, to understand that the BH has a temperature (i.e. Hawking temperature), as viewed by a stationary observer far away from the horizon, we will argue that the notion of a thermal state corresponds to the periodicity in imaginary time, that is, by finding the period in the imaginary time, one can infer the temperature of the state. The heuristic argument is as follows. Recall in quantum statistical mechanics, a thermal state is described by the partition function, which is defined as

$$Z = \sum_j \mathrm{e}^{-\beta E_j}, \tag{78}$$

where $\beta = 1/T$ $(k_B = 1)$ and E_j is the energy of the state $|j\rangle$, an eigenstate of the Hamiltonian. So we can rewrite the partition function as

$$Z = \sum_j \langle j|\mathrm{e}^{-\beta H}|j\rangle = \mathrm{Tr}\Big(\mathrm{e}^{-\beta H}\Big). \tag{79}$$

On the other hand, the propagator (2-point Green function) in quantum mechanics is

$$K(q',t;q,0) = \left\langle q'\middle|\mathrm{e}^{-iHt}\middle|q\right\rangle. \tag{80}$$

Suppose we want to recover the partition function in this formalism. We consider t to be complex parameter, and consider it to be purely imaginary, so we can write $t = -i\tau$, where τ is real[7]. Then

$$K(q',-i\tau;q,0) = \left\langle q'\middle|\mathrm{e}^{-H\tau}\middle|q\right\rangle = \sum_j \mathrm{e}^{-\tau E_j}\langle j|q\rangle\left\langle q'\middle|j\right\rangle. \tag{81}$$

[6]By definition, an observer's proper acceleration is the 3-acceleration measured in the comoving frame.
[7]This is equivalent to a Wick rotation, where we transform from Minkowski space to Euclidean space.

Setting $q' = q, \tau = \beta$ and integrating over q, we get exactly the partition function

$$\int \mathrm{d}q\, K(q, -i\beta; q, 0) = \sum_j \mathrm{e}^{-\beta E_j} \langle j| \int \mathrm{d}q\ |q\rangle \langle q||j\rangle = Z. \tag{82}$$

What this tells us is that a thermal state in statistical mechanics related to a quantum mechanical system that evolves in an imaginary time $t = -i\tau$, where τ is periodic: $\tau \sim \tau + \beta$. This observation holds true in QFT. To describe a system at finite temperature T, we analytically continue to the Euclidean signature, $t \to -i\tau$, and let τ to be periodic with period $\beta = 1/T$. Conversely, if the Euclidean continuation of a QFT is periodic in the imaginary time direction, we conclude that the QFT is at a finite temperature.

With this in mind, we may analytically continue the Schwarzschild metric to Euclidean signature with $t \to -i\tau$:

$$\mathrm{d}s_E^2 = f\mathrm{d}\tau^2 + \frac{1}{f}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2. \tag{83}$$

Near the horizon, the metric becomes

$$\mathrm{d}s_E^2 = \rho^2\kappa^2\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + r_s^2\mathrm{d}\Omega_2^2 = \rho^2\mathrm{d}\theta^2 + \mathrm{d}\rho^2 + r_s^2\mathrm{d}\Omega_2^2, \tag{84}$$

where

$$\theta \equiv \kappa\tau = \frac{\tau}{r_s}. \tag{85}$$

Note that the first two terms are just the polar coordinates in Euclidean $\mathcal{R}_2$. This metric has a conical singularity unless θ is periodic in 2π, i.e. $\theta \sim \theta + 2\pi$. Since the horizon is non-singular in Lorentzian signature, it should also not singular in Euclidean. Therefore, τ must be periodic,

$$\tau \sim \tau + \frac{2\pi}{\kappa}. \tag{86}$$

Since t is the proper for an observer at $r = \infty$, by the argument above, this observer O_∞ must feel a temperature

$$T = \frac{1}{\beta} = \frac{\kappa}{2\pi} = \frac{1}{8\pi GM}. \tag{87}$$

This is the famous Hawking temperature associated with the Hawking radiation. Similarly for the 2d Rindler spacetime,

$$\mathrm{d}s^2 = -\rho^2\mathrm{d}\eta^2 + \mathrm{d}\rho^2 \xrightarrow{\eta \to -i\theta} \mathrm{d}s_E^2 = \rho^2\mathrm{d}\theta^2 + \mathrm{d}\rho^2. \tag{88}$$

As shown before, a uniformly accelerating observer in the Rindler spacetime follows a trajectory of constant ρ, and so the local proper time of this observer is $\mathrm{d}\tau_{\mathrm{loc}}^2 = \rho^2\mathrm{d}\eta^2$. Since θ must be period in 2π, η must also be period in 2π, and therefore τ_{loc} must be period in $2\pi\rho$. So the Rindler observer feels a temperature

$$T_{\mathrm{loc}}^{\mathrm{Rindler}} = \frac{1}{2\pi\rho} = \frac{a}{2\pi}, \tag{89}$$

where we have used the relation $a = 1/\rho$ in Eq. (76). This temperature is known as the Unruh temperature, and is proportional to the observer's proper acceleration.